Week 4: Functions: injectivity, surjectivity, bijectivity

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups A&B, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups C&E, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups D&F, benoit.tran@polytechnique.edu).

1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Give an example of a function:

a) which is one-to-one but not onto; b) which is onto but not one-to-one;

c) which is bijective; d) which is neither one-to-one, nor onto.

Solution of exercise 1.

a) $f : \{1,2\} \to \{1,2,3\}$ with $f(x) = x$ for $x = 1,2$.

b) $f : \{1,2\} \to \{1\}$ with $f(x) = 1$ for $x = 1,2$.

c) $f : \{1,2\} \to \{1,2\}$ with $f(x) = x$ for $x = 1,2$.

d) $f : \{1,2\} \to \{1,2\}$ with $f(x) = 1$ for $x = 1,2$. □

Exercise 2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ (recall that $\mathbb{R}^2$ denotes the set of all ordered couples (x,y) with $x \in \mathbb{R}$ and $y \in \mathbb{R}$) be the function defined by $F(x,y) = (x+y, x-y)$ for every $(x,y) \in \mathbb{R}^2$. Is F a bijection?

Solution of exercise 2.

1st step: f is injection. Fix $(x,y),(x',y') \in \mathbb{R}^2$ such that $F(x,y) = F(x',y')$. Then

$$\begin{cases} x+y = x'+y' & (A) \\ x-y = x'-y'. & (B) \end{cases}$$

Now, $(A)+(B)$ gives $2x = 2x'$, *i.e.* $x = x'$. Substituting this into (A) gives $y = y'$. We have proved

$$(x,y) = (x',y').$$

Therefore F is one-to-one.

2d step: f is onto. Fix $(u,v) \in \mathbb{R}^2$. We want to find $(x,y) \in \mathbb{R}^2$ such that $F(x,y) = (u,v)$:

$$\begin{cases} x+y = u & (E) \\ x-y = v. & (F) \end{cases}$$

$(E)+(F)$ yields $2x = u+v$, *i.e.* $x = (u+v)/2$. Substituting this into (E) gives $(u+v)/2 + y = u$, *i.e.* $y = (u-v)/2$.

We finally check that indeed

$$F\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = (u,v).$$

Faire remarquer que quand on raisonne par implications, il faut vérifier réciproquement que les solutions obtenues conviennent bien (ou pas). □

Exercise 3. Let I be an interval and $f : I \to \mathbb{R}$ be an increasing function. Show that f is one-to-one.

Solution of exercise 3. We fix $x, y \in I$ such that $x \neq y$. We show that $f(x) \neq f(y)$.

First case: $x < y$. Since f is increasing, we have $f(x) < f(y)$. Therefore $f(x) \neq f(y)$.

Second case: $y < x$. Since f is increasing, we have $f(x) > f(y)$. Therefore $f(x) \neq f(y)$. □

Exercise 4. Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ be two functions.

1) Show that if $g \circ f$ is one-to-one, then f is one-to-one.

2) Show that if $g \circ f$ is onto, then g is onto.

Solution of exercise 4.

1) Assume that $g \circ f$ is one-to-one. Fix $x, y \in A$ such that $f(x) = f(y)$. Then $g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y)$. Since $g \circ f$ is one-to-one, this implies that $x = y$. Hence f is one-to-one.

2) Assume that $g \circ f$ is onto. Fix $c \in C$. Since $g \circ f$ is onto, there exists $a \in A$ such that $g \circ f(a) = c$. Hence $g(f(a)) = c$, so that $f(a)$ is a pre-image of c by g. Hence g is onto.

□

Definition.

Let $f : X \to Y$ be a bijection. Recall that we define the function $f^{-1} : Y \to X$, called the inverse (bijection) of f, as follows. Fix $y \in Y$. Let x be the unique pre-image of y by f, and set $f^{-1}(y) = x$.

Exercise 5. Let $f : X \to Y$ be a bijection. Show that:

a) $\forall x \in X$, $f^{-1} \circ f(x) = x$

b) $\forall y \in Y, f \circ f^{-1}(y) = y$.

Solution of exercise 5.

a) By definition, $f^{-1}(f(x))$ is the unique preimage of $f(x)$, which is indeed x.

b) Fix $y \in Y$ and let $x \in X$ be the preimage of y by f, so that $y = f(x)$. Then $f \circ f^{-1}(y) = f \circ f^{-1}(f(x)) = f(x)$ by a). Since $f(x) = y$, we indeed have $f \circ f^{-1}(y) = y$. □

Exercise 6. Let $f : [0, +\infty) \to \mathbb{R}_+$ be the function defined by $f(x) = (\sqrt{x^2+1} + 2)^2$ for every $x \geq 0$. Show that f is a bijection between $[0, +\infty)$ and $[9, +\infty)$, and give a simple expression of its inverse bijection.

Solution of exercise 6. The function f is increasing (as a composition of increasing functions), so f is one-to-one.

We now prove that f is onto. Let u be an arbitrary number in $[9,+\infty)$. We shall find $x \in [0,+\infty)$ such that

$$u = f(x) = \left(\sqrt{x^2+1}+2\right)^2.$$

Solving this equation gives

$$x = \sqrt{\Bigg(\underbrace{\sqrt{u}-2}_{\in[1,+\infty)}\Bigg)^2 - 1} \in [0,+\infty).$$

Conversely, we can check that

$$f\left(\sqrt{\left(\sqrt{u}-2\right)^2-1}\right) = u.$$

This proves that f is onto, hence a bijection, and that $f^{-1}(u) = \sqrt{\left(\sqrt{u}-2\right)^2-1}$ for $u \geq 9$. □

2 Homework exercises

You have to individually hand in the written solution of the next exercises to your TA on October, 21th.

Exercise 7. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by $f(x) = x^2+4x+4$ for every $x \geq 0$.

a) Prove that f is bijection between $[0,+\infty)$ and $[4,+\infty)$.

b) Give a simple expression of its inverse.

Solution of exercise 7. The function f is increasing as a sum of increasing functions, so f is one-to-one (it is also possible to check by hand by using the computations that follow that if $f(x) = f(x')$ for $x, x' \geq 0$, then $x = x'$).

We now show that f is onto. Fix $t \in [4,+\infty)$. We aim at finding $x \in [0,+\infty)$ such that

$$f(x) = x^2+4x+4 = t. \qquad (\star)$$

We have to solve this equation, where x is the unknown variable and t is a parameter. The discriminant is given by

$$\Delta = 4^2 - 4(4-t) = t^2 \geq 0.$$

We deduce that equation ($\star$) has two solutions:

$$x_1(t) = -2+\sqrt{t} \qquad \text{and} \qquad x_2(t) = -2-\sqrt{t}.$$

We discard the solution $x_2(t)$ since $x_2(t) < 0$. The solution $x_1(t)$ satisfies $x_1(t) \geq -2+\sqrt{4} = 0$. By construction, we have $f(x_1(t)) = t$: for every $t \geq 1$,

$$f(-2+\sqrt{t}) = t.$$

This proves that f is a bijection between $[0,+\infty)$ and $[4,+\infty)$ and that $f^{-1}(t) = -2+\sqrt{t}$ for every $t \geq 4$. □

Exercise 8. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Show that if $g \circ f$ is one-to-one and f is onto, then g is one-to-one. Is the converse always true? Justify your answer.

Solution of exercise 8. Fix $a, b \in Y$ such that $g(a) = g(b)$. Since f is onto, there exists $a', b' \in X$ such that $a = f(a')$ and $b = f(b')$. Hence $g(f(a')) = g(f(b'))$. Since $g \circ f$ is one-to-one, if follows that $a = b$. Therefore g is one-to-one.

The converse is not true: take $X = Y = Z = \{1,2\}$ and define $f(1) = 1$, $f(2) = 1$, $g(1) = 1$ and $g(2) = 2$. Then $g \circ f$ is not one-to-one (because $g \circ f(1) = g \circ f(2)$, and f is not onto (since 2 has no preimage by f). □

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 4.

Exercise 9. Let $n \geq 2$ and $k \geq 2$ be integers.

a) How many functions $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,k\}$ can one define?

b) How many functions $f : \{1,2,\ldots,n\} \to \{1,2\}$ are onto?

c) How many functions $f : \{1,2,3\} \to \{1,2,\ldots,k\}$ are one-to-one?

Solution of exercise 9.

a) There are k choices for every $f(x)$ with $1 \leq x \leq n$, so the answer is k^n.

b) Every such function is onto, excepted the constant function equal to 1 and the constant function equal to 2. The answer is thus $2^n - 2$.

c) For $k \leq 2$, by the pigeonhole principle there are no such functions. If $k \geq 3$, there are k choices for $f(1)$, $k-1$ choices for $f(2)$, $k-2$ choices for $f(3)$. The answer is $k(k-1)(k-2)$ if $k \geq 3$, and 0 otherwise. □

Exercise 10. Set $\mathbb{N} = \{1,2,\ldots\}$ and recall that $\mathbb{N}^2$ denotes the set of all ordered couples (x,y) with $x \in \mathbb{N}$ and $y \in \mathbb{N}$. Let $\phi : \mathbb{N}^2 \to \mathbb{N}$ be defined by $\phi(u,v) = 2^{u-1} \times (2v-1)$ for $u, v \in \mathbb{N}$. Prove that ϕ is a bijection.

(Hint: You can use the fact that any integer can be represented in exactly one way, up to the order of the factors, as a product of prime powers.)

Solution of exercise 10.

Surjectivity. Fix $n \in N$. By factoring n into prime powers, we can write $n = 2^a b$ with $a \geq 0$ and b odd. Therefore we can write $b = 2k-1$ with $k \geq 1$, and $\phi((a+1,k)) = n$.

Injectivity. Fix $(a,b) \in \mathbb{N}^2$ and $(u,v) \in \mathbb{N}^2$ such that $\phi(a,b) = \phi(u,v)$. Then $2^{u-1}(2v-1) = 2^{a-1}(2b-$

1). By symmetry, without loss of generality, we may assume that $u \geq a$. Then $2v-1 = 2^{a-u}(2b-1)$. Since $2v-1$ is odd, we must have $2^{a-u} = 1$, so that $a = u$. In turn, this implies that $2v-1 = 2b-1$, so that $b = v$. Hence $(a,b) = (u,v)$, which shows that ϕ is one-to-one. □

Exercise 11. Recall that for a given point $M = (a,b)$ in the plane, the coordinates of the symmetric point to M with respect to the straight line with equation $y = x$ are (b,a).

Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection. Prove that the two graphical representations of f and f^{-1} in the plane are symmetric with respect to the straight line $\{y = x\}$.

Solution of exercise 11. The graphical representation of a function $g : \mathbb{R} \to \mathbb{R}$ is the set of points $\{(x,g(x)); x \in \mathbb{R}\}$. Therefore, graphical representation of f is the set of points $A = \{(x,f^{-1}(x)); x \in \mathbb{R}\}$ and that of f^{-1} is the set of points $B = \{(x,f(x)); x \in \mathbb{R}\}$. The symmetric of B with respect to the straight line $y = x$ is therefore the set $B' = \{(f(x),x); x \in \mathbb{R}\}$. We show that $A = B'$ by double inclusion.

Take $(x,f^{-1}(x)) \in A$ with $x \in \mathbb{R}$. Since f is onto, we can write $x = f(y)$ with $y \in \mathbb{R}$. Then $(x,f^{-1}(x)) = (f(y),f^{-1}(f(y))) = (f(y),y) \in B'$.

Take $(f(x),x) \in B'$ with $x \in \mathbb{R}$. Since f^{-1} is onto, we can write $x = f^{-1}(y)$ with $y \in \mathbb{R}$. Then $(f(x),x) = (f(f^{-1}(y)),f^{-1}(y)) = (y,f^{-1}(y)) \in A$. □

Exercise 12. Let E and F be two non-empty sets. Show that the following two assertions are equivalent:

(i) there exists an injection from E to F,

(ii) there exists a surjection from F to E.

Solution of exercise 12.

We first show that (i) implies (ii). Let $f : E \to F$ be an injection and fix $x_0 \in E$. We define $g : F \to E$ as follows. For $u \in F$:

Case 1: there exists $x \in E$ such that $f(x) = u$. Then we define $g(u) = x$ (this is well defined, since by injectivity of f we cannot find $y \neq x$ such that $f(y) = u$).

Case 2: otherwise, set $g(u) = x_0$.

Then g is a onto, since $g(f(x)) = x$ for every $x \in E$ by construction.

We next show that (ii) implies (i). Let $f : F \to E$ be a surjection. We define $g : E \to F$ as follows. For $u \in E$, choose one element $x \in F$ such that $f(x) = u$ (this element is not necessarily unique, but exists by surjectivity of f), and define $g(u) = x$. Let us now show that g is one-to-one. Assume that $g(u) = g(v)$ with $u,v \in E$. Then, by construction of f, $f(g(v)) = u$ and $f(g(u)) = v$. But $f(g(u)) = f(g(v))$, so that $u = v$. Therefore g is an injection. □

Exercise 13. If A and B are two sets, we denote by A^B the set of all functions from B to A.

a) Let E,F,G be sets with $E \neq \emptyset$, and $f : F \to G$ a function. Show that f is one-to-one if and only if

$$\forall g,h \in F^E, \quad f \circ g = f \circ h \implies g = h.$$

b) Let F,G,H be sets such that H has at least two different elements, and $f : F \to G$ a function. Show that f is onto if and only if

$$\forall g,h \in H^G, \quad g \circ f = h \circ f \implies g = h.$$

Solution of exercise 13.
a) First assume that f is one-to-one. Fix $g,h \in F^E$. Then, for every $x \in E$, $f(g(x)) = f(h(x))$, and since f is one-to-one, $g(x) = h(x)$. Therefore $g = h$.

Now assume that for every $g,h \in F^E$, $g \circ f = h \circ f \implies g = h$. Fix $y_1, y_2 \in F$ such that $f(y_1) = f(y_2)$. We shall show that $y_1 = y_2$. To this end, consider the (constant) function $g : E \to F$ defined by $g(x) = y_1$ for every $x \in E$ and the (constant) function $h : E \to F$ defined by $h(x) = y_2$ for every $x \in E$. Then for every $x \in E$:

$$f \circ g(x) = f(y_1) = f(y_2) = f \circ h(x).$$

Therefore $g = h$. Since $E \neq \emptyset$, we may choose $x_0 \in E$, so that $y_1 = g(x_0) = h(x_0) = y_2$. This shows that f is one-to-one.

b) First assume that f is onto. Fix $g,h \in H^G$ such that such that $g \circ f = h \circ f$. Take any $y \in G$. Then there exists $x \in F$ such that $y = f(x)$. Therefore

$$g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y).$$

This shows that $g = h$.

Now assume that for every $g,h \in H^G$, $g \circ f = h \circ f \implies g = h$. Since H has at least two elements, we can choose $z_1, z_2 \in H$ such that $z_1 \neq z_2$. Now consider the (constant) function $g : G \to H$ defined by $g(x) = z_1$ for every $x \in G$ and the function $h : G \to H$ defined by

$$\forall y \in G, \quad h(y) = \begin{cases} z_1 & \text{if there exists } x \in F \text{ such that } y = f(x) \\ z_2 & \text{otherwise.} \end{cases}$$

Then for every $x \in F$, $(g \circ f)(x) = g(f(x)) = z_1$ and $(h \circ f)(x) = h(f(x)) = z_1$. Therefore $g \circ f = h \circ f$. This implies that $g = h$. Since g cannot take the value z_2, this implies that h cannot take the value z_2. Therefore $h(y) = z_1$ for every $y \in G$, which implies by definition of h that for every $y \in G$ there exists $x \in F$ such that $y = f(x)$. This shows that h is onto. □

4 Fun exercises (optional)

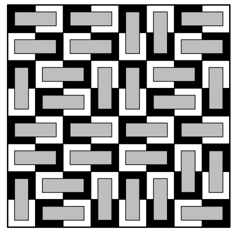
The solution of these exercises will be available on the course webpage at the end of week 4.

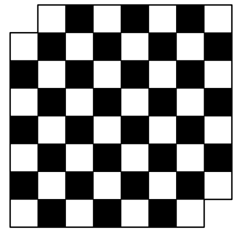
Exercise 14. Does there exist a bijection between $(0,1)$ and $[0,1]$?

Solution of exercise 14. Yes! Here is an example of a bijection $f : (0,1) \to [0,1]$. First consider the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$ Define f to map every other point that is not in this sequence to itself, and map the above sequence to the corresponding points in the following one: $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ In other words, map $\frac{1}{2}$ to 0, $\frac{1}{3}$ to 1, and then map $\frac{1}{n}$ to $\frac{1}{n-2}$ for every $n \geq 4$.

It is then a simple matter to check that f is bijective. □

Exercise 15. It is clearly possible to cover an 8×8 chessboard with 32 dominos of size 2×1 (see the left picture below). Is it possible cover the chessboard on the right (in which two diagonally opposite corners have been removed) with 31 dominos?





Solution of exercise 15. Every domino covers one white square and one black square. As there are only 30 black squares, it is impossible to cover the chessboard with 31 dominos. □