Week 2, September 30: Formal logic, truth tables, methods of proof

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. If $x, y \in \mathbb{R}$ are real numbers, write the negation of the following assertions:

a) $x = 0$ b) $x \geq 0$ c) $|x| < 1$ d) ($x \geq 0$ and $y \geq 0$) or ($x \leq 0$ and $y \leq 0$).

Solution of exercise 1.

a) $x \neq 0$;

b) $x < 0$;

c) $x \geq 1$ or $x \leq -1$.

d) The assertion is

$$\Big((x \geq 0) \wedge (y \geq 0)\Big) \vee \Big((x \leq 0) \wedge (y \leq 0)\Big).$$

Its negation is

$$\Big((x < 0) \vee (y < 0)\Big) \wedge \Big((x < 0) \vee (y < 0)\Big),$$

that is: ($x < 0$ or $y < 0$) and ($x < 0$ or $y < 0$).

Remark. The use of and, or in "text mode" has to be as explicit as possible so that there is no ambiguity concerning the parenthesis. □

Exercise 2. When Y, Z are sets, write the contrapositive of $(x \in Y) \implies (x \in Z) \vee (x \notin Y)$.

Solution of exercise 2. The contrapositive is

$$(x \notin Z) \wedge (x \in Y) \implies x \notin Y.$$

□

Exercise 3. In each case, are the two assertions logically equivalent?

a) $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$ b) $P \vee Q$ and $(P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$.

Solution of exercise 3.

a) One can show that they are logically equivalent by writing a truth table:

P	Q	R	$P \implies R$	$Q \implies R$	$(P \implies R) \wedge (Q \implies R)$	$P \vee Q$	$(P \vee Q) \implies R$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

b) They are not logically equivalent: when P and Q are both true, $P \vee Q$ is true while $(P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$ is false.

□

Exercise 4. Which of the following statements are true? Justify your answer.

a) $(\neg(\emptyset = \emptyset)) \wedge (\neg(\emptyset \in \emptyset))$ b) $(\emptyset \in \emptyset) \implies (\emptyset \subseteq \emptyset)$ c) $(\emptyset = \emptyset) \vee (\neg(\emptyset \in \emptyset))$.

Solution of exercise 4.

a) $\neg(\emptyset = \emptyset)$ is false, so the statement is false (because of the $\wedge$ symbol).

b) $\emptyset \in \emptyset$ is false, so the statement is true (by definition, $A \implies B$ is always true when A is false).

c) $\emptyset = \emptyset$ is true, so the statement is true (because of the $\vee$ symbol). □

Exercise 5. If A, B are subsets of a set E, write the negation of the assertion "$x \in \overline{A} \cap B$" in the form "$x \in \ldots$".

Solution of exercise 5. If C is a subset of E, the negation of $x \in C$ is $x \notin C$, which is the same as $x \in \overline{C}$ (where $\overline{U}$ denotes the complement U in E).

Hence the negation of $x \in \overline{A} \cap B$ is $x \in \overline{\overline{A} \cap B}$. But by results of Exercise Sheet 1, $\overline{\overline{A} \cap B} = \overline{\overline{A}} \cup \overline{B} = A \cup \overline{B}$. We conclude that the the negation of $x \in \overline{A} \cap B$ is $x \in A \cup \overline{B}$. □

Exercise 6. When P and Q are assertions, write a logical formula for the following phrases:

a) P but not Q b) either P or Q but not both c) P, unless Q.

Also give their negations.

Solution of exercise 6.

a) $P \wedge (\neg Q)$, whose negation is $(\neg P) \vee Q$.

b) $(P \vee Q) \wedge (\neg(P \wedge Q))$, whose negation is $((\neg P) \wedge (\neg Q)) \vee (P \wedge Q)$.

c) This term "unless" demands interpretation. One plausible solution, is to interpret it as P has to be true, except when Q is true, in which case we do not ask anything about P. This gives $P \vee Q$. Its negation is $(\neg P) \wedge (\neg Q)$. □

Exercise 7. Write the following statement in if-then form: "Every perfect integer is even" (you do not need to know what a "perfect" integer is to solve the exercise).

Solution of exercise 7. If x is a perfect integer, then x is even.

Remark. An integer is perfect if it is equal to the sum of its proper (meaning that the number itself is excluded) positive divisors. It is not known if this statement is true or false!!! Indeed, it is not known if there exists an odd perfect number (but for example it is known - Ochem & Rao 2002 - that any odd perfect number has at least 101 not necessarily distinct prime factors.) □

Exercise 8. Let x and y be two integers. Show that $x+y$ is even if and only if x and y have the same parity.

Solution of exercise 8.

We argue by double implication.

First observe that the converse implication is easy. Indeed, if x and y have the same parity, then $x+y$ is indeed even.

We now show the direct implication. Its contrapositive

"If x and y are two integers with opposite parity, then their sum must be odd."

So we assume x and y have opposite parity. Since one of these integers is even and the other odd, there is no loss of generality to suppose x is even and y is odd. Thus, there are integers k and m for which $x = 2k$ and $y = 2m+1$. Now then, we compute the sum $x+y = 2k+2m+1 = 2(k+m)+1$, which is an odd integer by definition. □

Exercise 9. Around 546 BC, Croesus (595 BC – 546 BC) began preparing a campaign against Cyrus the Great of Persia. He turned to the Delphic oracle for advice. The oracle said that if Croesus attacked the Persians, he would destroy a great empire. Croesus attacked the Persians, but was eventually defeated by Cyrus. Was the oracle wrong?

Solution of exercise 9. The oracle was not wrong. Croesus indeed destroyed a great empire: his own!

In order to avoid misunderstandings, one should be precise! □

Exercise 10. During the first world war, someone noticed that the planes that had returned to base had impacts everywhere, except on the cockpit. She deduced that the cockpit was fragile and had to be reinforced. Do you agree?

Solution of exercise 10. Let A and B be the following assertions

$$A\text{: "the cockpit does not have impacts"}, \qquad B\text{: "the plane returns to base".}$$

The observation tells us that if B is true, then A is true (but that we do not know what happens when B is false). Therefore, if A is false, then B is false (indeed, if B was true, then A should be true and this would be a contradiction). This means that if the cockpit has impacts, then the plane does not return to base, which indeed indicates that the cockpit is fragile. □

Exercise 11. What do you think of the following argumentation?

"All impartial observers and all credible theorists hold that when the basic structures of a society are fair, citizens conform to them of their own will. The fact that citizens in our societies do not rebel thus constitutes a powerful and convincing proof of the justice of our basic institutions, and all our so-called revolutionaries would be wise to think about that carefully."

Solution of exercise 11. Let A and B be the following assertions:

A: "the basic structures of a society are fair", B: "citizens do not rebel".

The first sentence states that if A is true, then B is true. The second sentence deduces that if B is true, then A is true. This reasoning is not correct, since $A \implies B$ and $B \implies A$ are not logically equivalent.

Remark. However, it is known that $A \implies B$ and $\neg B \implies \neg A$ are logically equivalent. Therefore, assuming that the first sentence is true, it would be correct to deduce that if citizens rebel, then the basic structures of a society are unfair.

□

2 Homework exercises

You have to individually hand in the written solutions of the next two exercises to your TA on October, 7th.

Exercise 12. Let a, b be two positive integers.

1) Show that $a^2 > b^2 + 1 \implies a \geq b + 1$.

2) Is the implication $a^2 > b^2 + 1 \implies a \geq b + 1$ always true if a, b are integers (not necessarily positive)?

Solution of exercise 12.

1) We argue by contraposition and assume that $a \geq b + 1$ is not true. This implies that $a < b + 1$ and thus $a \leq b$ since a and b are integers. Then $a^2 \leq b^2$, and hence $a^2 \leq b^2 + 1$. We have thus shown that $a < b + 1 \implies a^2 \leq b^2 + 1$, which completes the proof.

2) No: if $a = -2$ and $b = 0$, we have $a^2 > b^2 + 1$ but $a \geq b + 1$ is not true.

□

Exercise 13. Let A and B be two sets. Show that $A \subseteq B \iff A \cap B = A$.

Solution of exercise 13. We argue by double implication.

∗ We first show that $A \subseteq B \implies A \cap B = A$. Assume that $A \subseteq B$. To show that $A \cap B = A$ we argue by double inclusion.

First, it is clear that $A \cap B \subseteq A$.

Second, take $x \in A$. Since $A \subseteq B$, we also have $x \in B$. Therefore $x \in A \cap B$. Hence $A \cap B \subseteq A$.

∗ We now show that $A \cap B = A \implies A \subseteq B$. Assume that $A \cap B = A$. Since $A \cap B \subseteq B$, we indeed have $A \subseteq B$.

This completes the proof. □

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 2.

Exercise 14. Below are 5 statements (in "formal logic" language). How many of them can be true together?

(a): if (b) is true then (a) is false.

(b): if number of the true statements is (strictly) greater than 2, then one of them is (c).

(c): at least one of (a) and (d) is false.

(d): (b) and (c) are both true or both false.

(e): (b) is true or false.

Solution of exercise 14.

$*$ (a) cannot be false. Indeed, assume by contradiction that (a) is false. This implies (by definition of the truth table of $\implies$) that (b) is true and that "(a) is false" is false, that is (a) is true. This is a contradiction. Therefore, it is impossible for (a) to be false and (a) is true.

$*$ Also, (e) is always true. So at least (a) and (e) are always true.

$*$ Since (a) is true, by contrapositive it follows that (b) is false. This implies (by definition of the truth table of $\implies$) that the number of the true statements is strictly greater than 2, but (c) is not one of them. Therefore (c) is false.

$*$ Since (b) and (c) are both false, then (d) is true (which is our third true statement).

Conversely, if (a), (d), (e) are true and (b), (c) are false there is no contradiction. So the answer is: 3 statements are true.

Remark. Rigorously speaking, this problem is ill-posed because the first assertion contains a self-reference. Indeed, first order predicate logic does not allow self-reference (which leads to various issues, such as the barber paradox). □

Exercise 15. Define the median of a set of k numbers as follows: first put the numbers in non-decreasing order; then the median is the middle number if k is odd, and the average of the two middle numbers if k is even. (For example, the median of $\{1,3,4,8,9\}$ is 4, and the median of $\{1,3,4,7,8,9\}$ is $(4+7)/2 = 5.5$.)

Let $n \geq 1$ be a positive integer, and define $S_n = \{1,2,\ldots,n\}$. If T is a non-empty subset of S_n, we say that T is balanced if the median of T is equal to the average of T. For example, for $n = 9$, each of the subsets $\{7\},\{2,5\},\{2,3,4\},\{5,6,8,9\}$, and $\{1,4,5,7,8\}$ is balanced; however, the subsets $\{2,4,5\}$ and $\{1,2,3,5\}$ are not balanced.

Prove that the number of balanced subsets of S_n is odd.

Solution of exercise 15. Consider the map $f : \mathcal{P}(S_n) \to \mathcal{P}(S_n)$ that takes a subset T of S_n and replaces every number $k \in T$ with $n+1-k$. In other words, it's a reversal map. In S_9, we have $f(\{1,2,4\}) = \{9,8,6\}$.

To simplify notation, denote by $A(T)$ the average of a set T and by $M(T)$ the median of a set T. By definition of f, for every $A \in \mathcal{P}(S_n)$:

$$A(T) = n+1-A(f(T)), \qquad M(T) = n+1-M(f(T)).$$

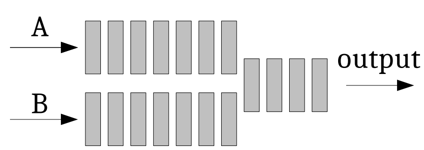
This implies that the function f has the property that it takes unbalanced sets to unbalanced sets, and balanced sets to balanced sets. Indeed $A(T) = M(T)$ if and only if $A(f(T)) = M(f(T))$. Also, if T is unbalanced, then $f(T) \neq T$. To show this property, we argue by contradiction and assume that $T = f(T)$. Then $A(T) = n+1-A(T)$ and $M(T) = n+1-M(T)$, which shows that $A(T) = M(T) = \frac{n+1}{2}$. Therefore T is balanced, which is a contradiction. Since $f(f(T)) = T$ for every set T, one may regroup the unbalanced subsets into pairs (made by an unbalanced set and its image by f), so they are even in number.

Let U be the number of unbalanced subsets and B the number of balances subsets. The previous paragraph shows that U is even. There are $2^n - 1$ non-empty subsets of S_n, which is an odd integer. Since $U + B = 2^n - 1$, this implies that B is odd. In other words, there is an odd number of balanced subsets. □

4 Fun exercises (optional)

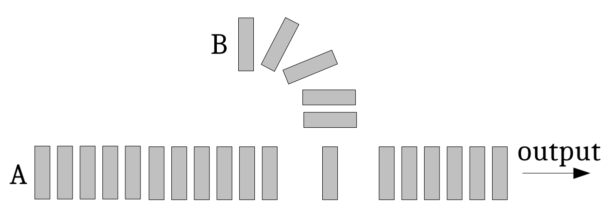
The solution of this exercise will be available on the course webpage at the end of week 2.

Exercise 16. **A computer with dominos.** A logic gate is the most elementary block of a digital circuit. For instance, the "or" gate passes on a signal if at least one of the inputs is on. It is possible to design logic gates with dominos: the gate passes on the signal if the chain of dominos is falling. With dominos the "or" gate is obtained by:



Can you make the gate "A and (not B)"? And the "and" gate?

Solution of exercise 16. For "A and (not B)",



For the AND gate, you can see a solution (and more about domino computers) at http://thescienceexplorer.com/technology/watch-computer-made-out-dominoes-do-basic-math. □