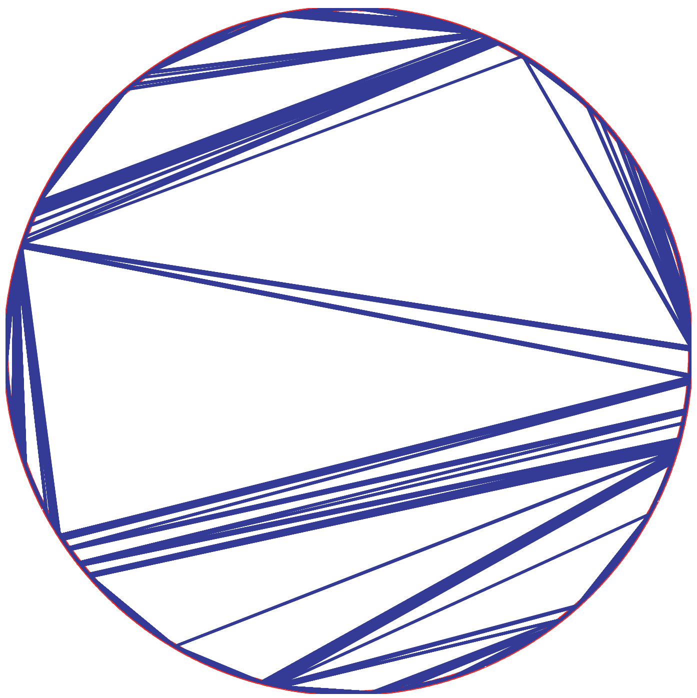
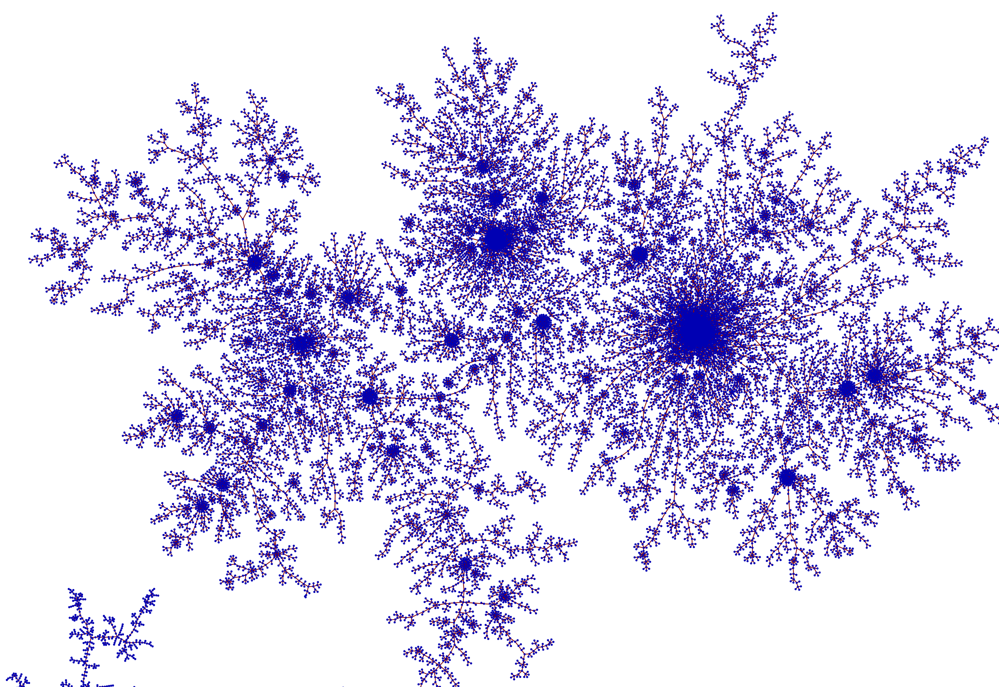
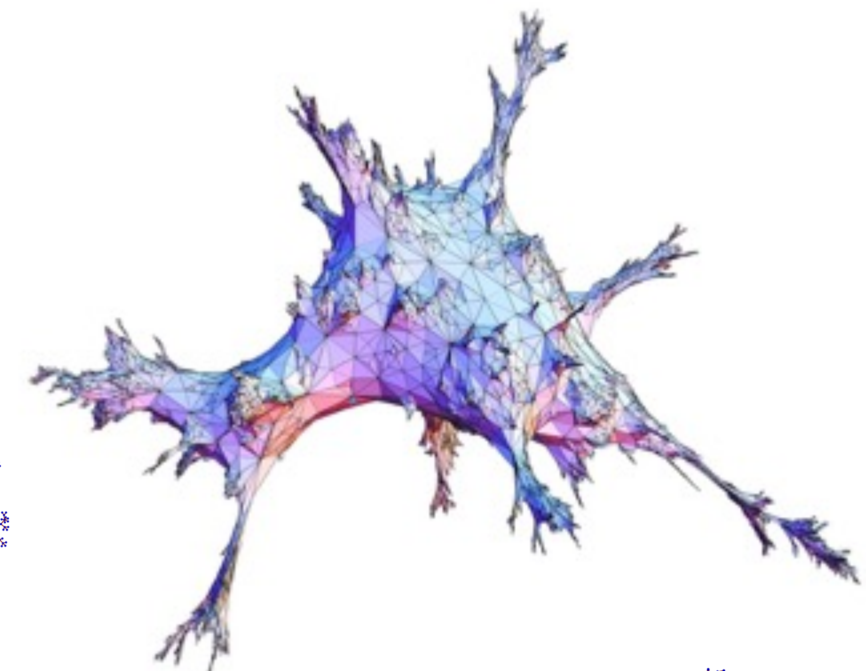
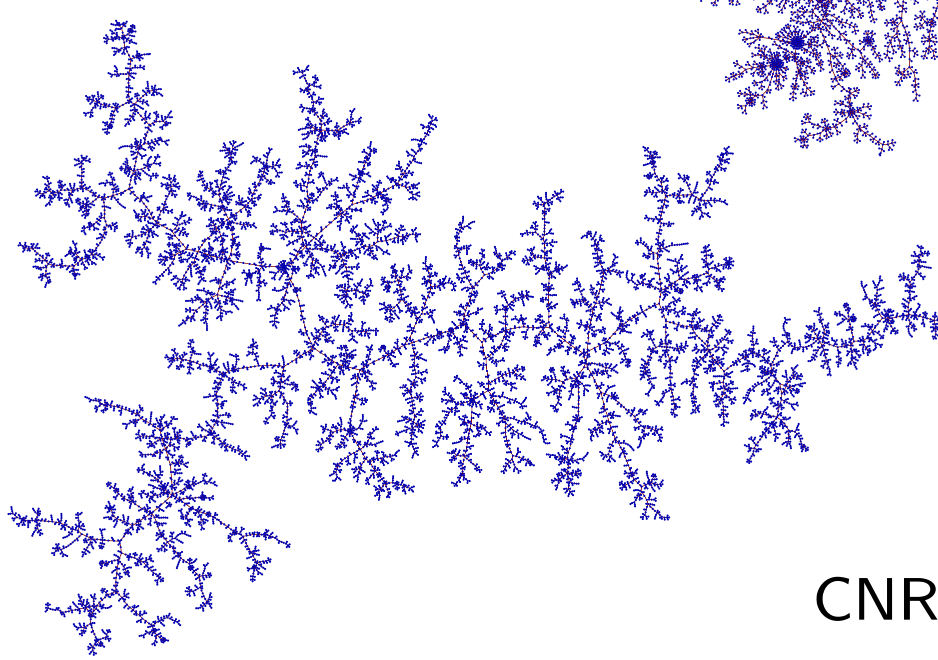
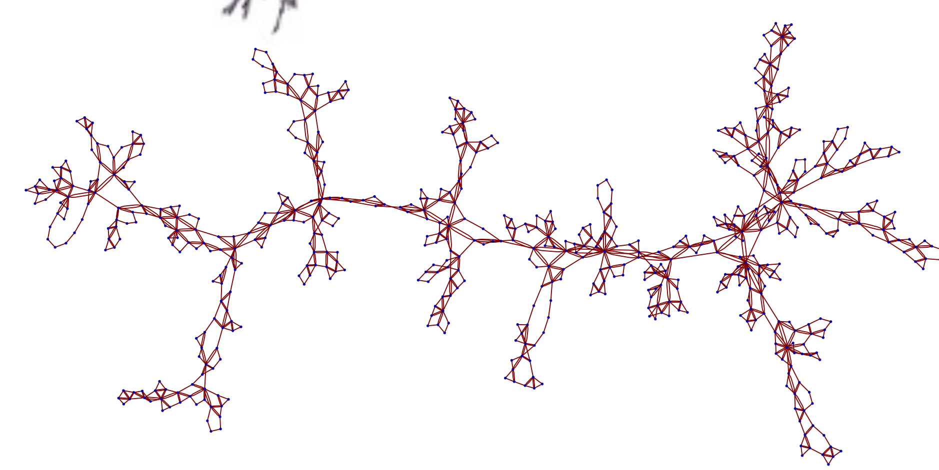
Scaling limits of large random discret structures











Igor Kortchemski

CNRS & École polytechnique

Lévy 2016 – Summer school on Lévy processes

Motivation for studying scaling limits

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- Find the cardinal of $\mathcal{X}_n$. (bijective methods, generating functions)
- Understand the typical properties of $\mathcal{X}_n$. Let X_n be an element of $\mathcal{X}_n$ chosen *uniformly at random.* What can be said of X_n?
- A possibility to study X_n is to find a continuous object X such that $X_n \to X$ as $n \to \infty$.

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- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards X, then X_n and Y_n share approximately the same properties for n large.

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- *In what space do the objects live?* Here, a metric space (Z, d) (complete separable).
- *What is the sense of the convergence when the objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[F(X)\right]$$

for every continous bounded function $F : Z \to \mathbb{R}$.

Outline

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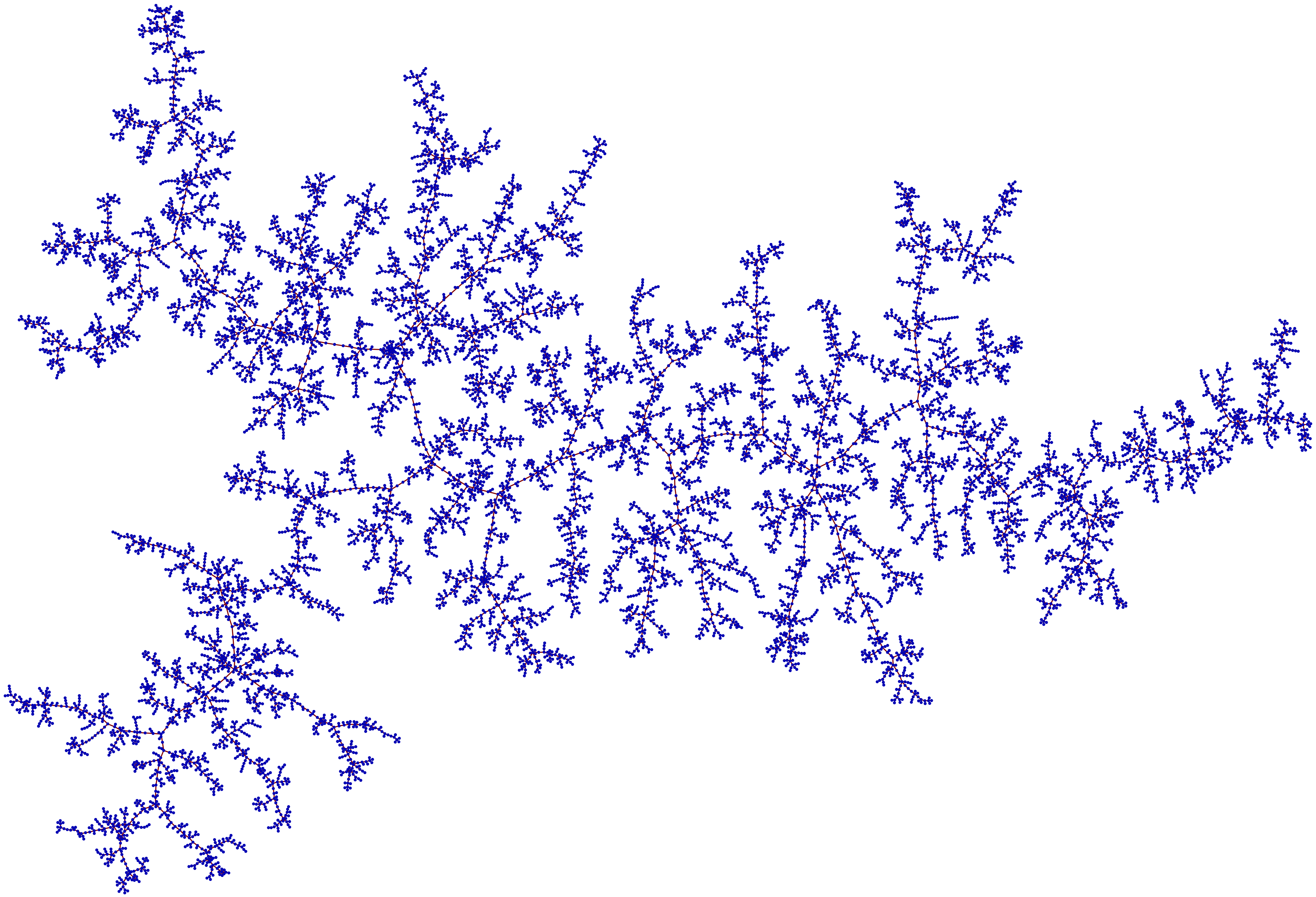
What does a large Bienaymé–Galton–Watson look like ?

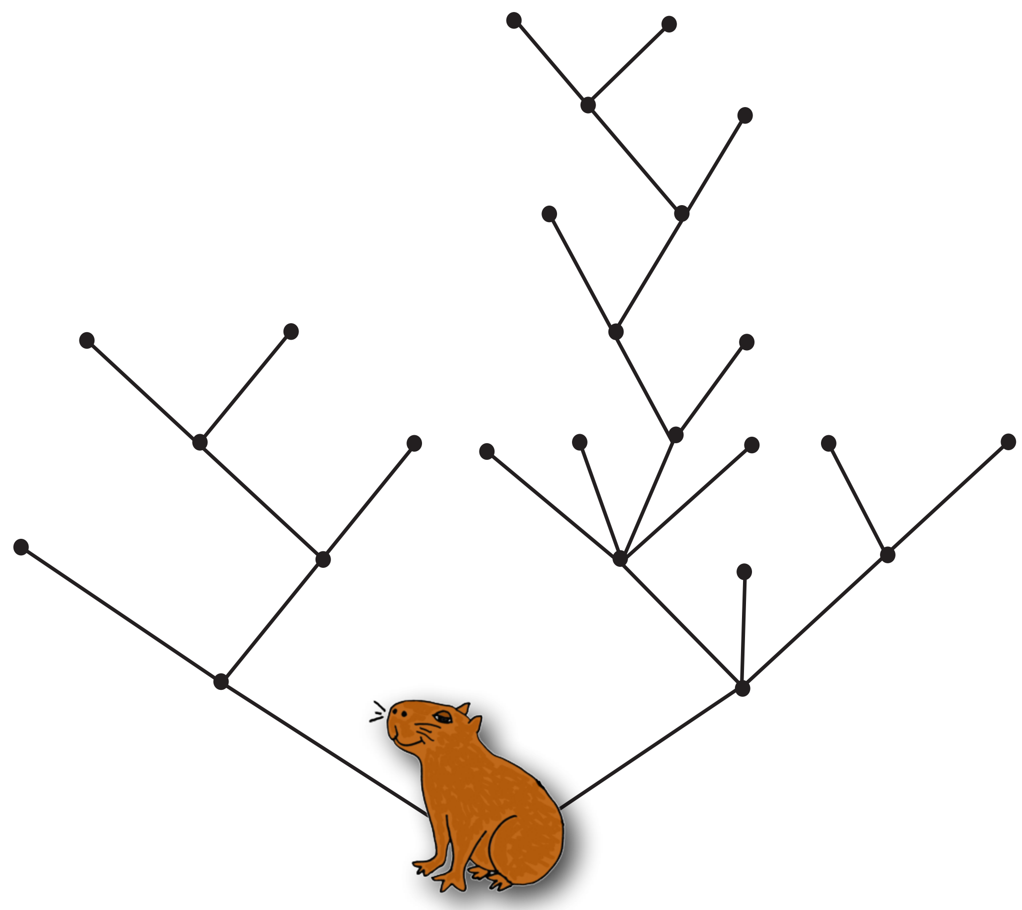
A simulation of a large random critical GW tree

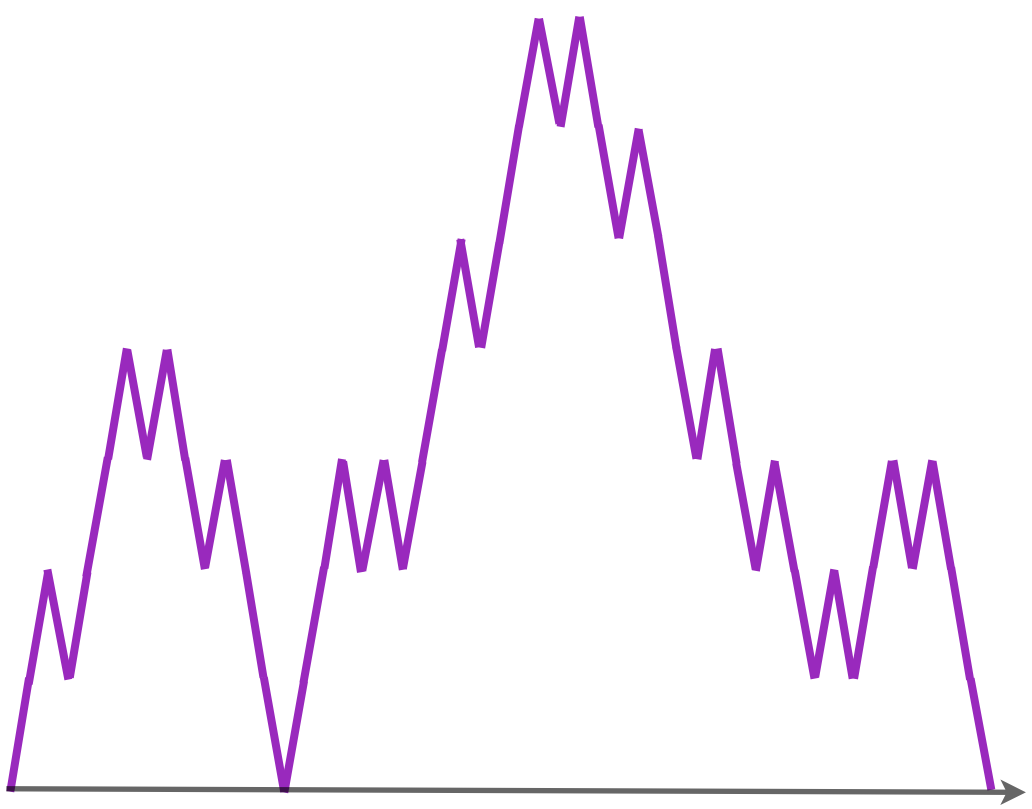


Coding trees by functions

Contour function of a tree

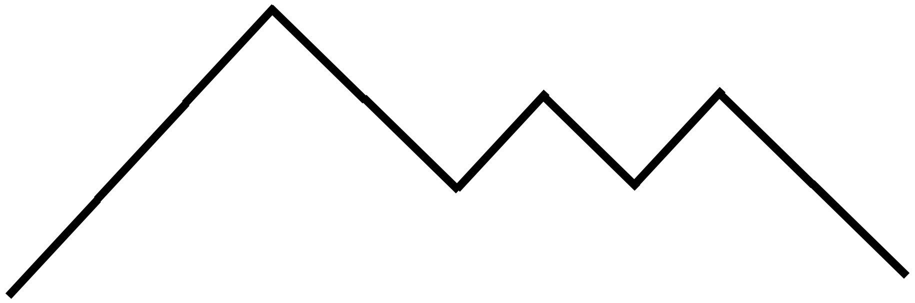
Define the contour function of a tree:





Coding trees by contour functions

Knowing the contour function, it is easy to recover the tree.



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Let σ^2 be the variance of μ. Let $t \mapsto C_t(\mathcal{T}_n)$ be the contour function of $\mathcal{T}_n$. Then:

$$\left(\frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n)\right)_{0\leqslant t\leqslant 1} \quad \xrightarrow[n\to\infty]{(d)}$$

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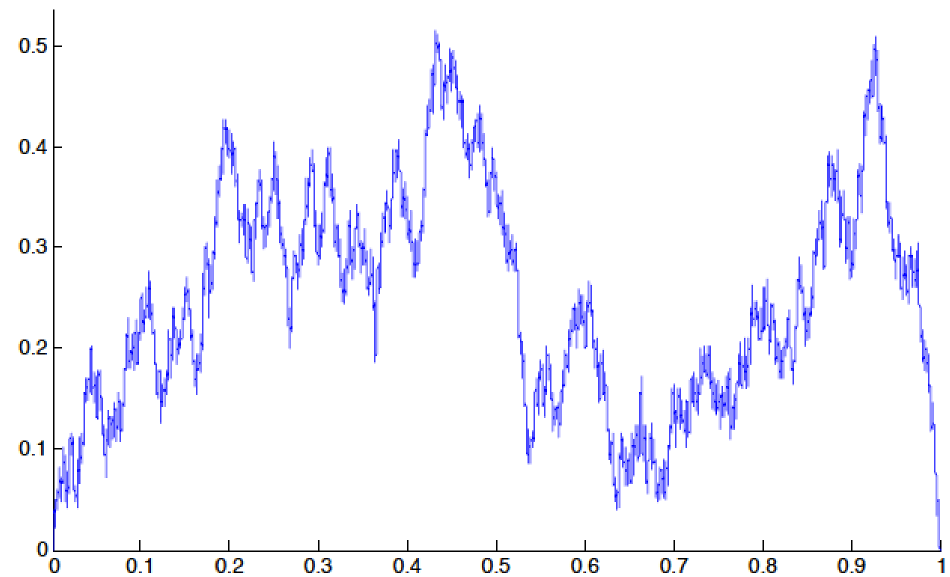

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- Go from the Lukasieiwicz path of $\mathcal{T}_n$ to its contour function.

Do the discrete trees converge to a continuous tree?

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Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!

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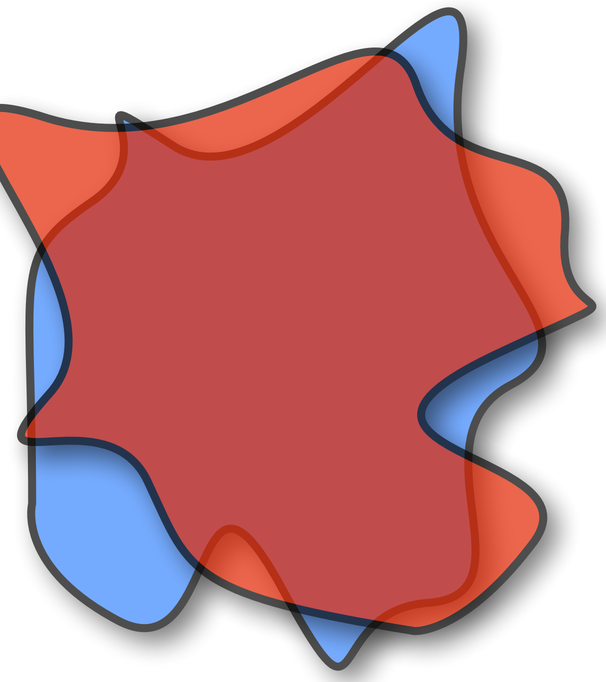

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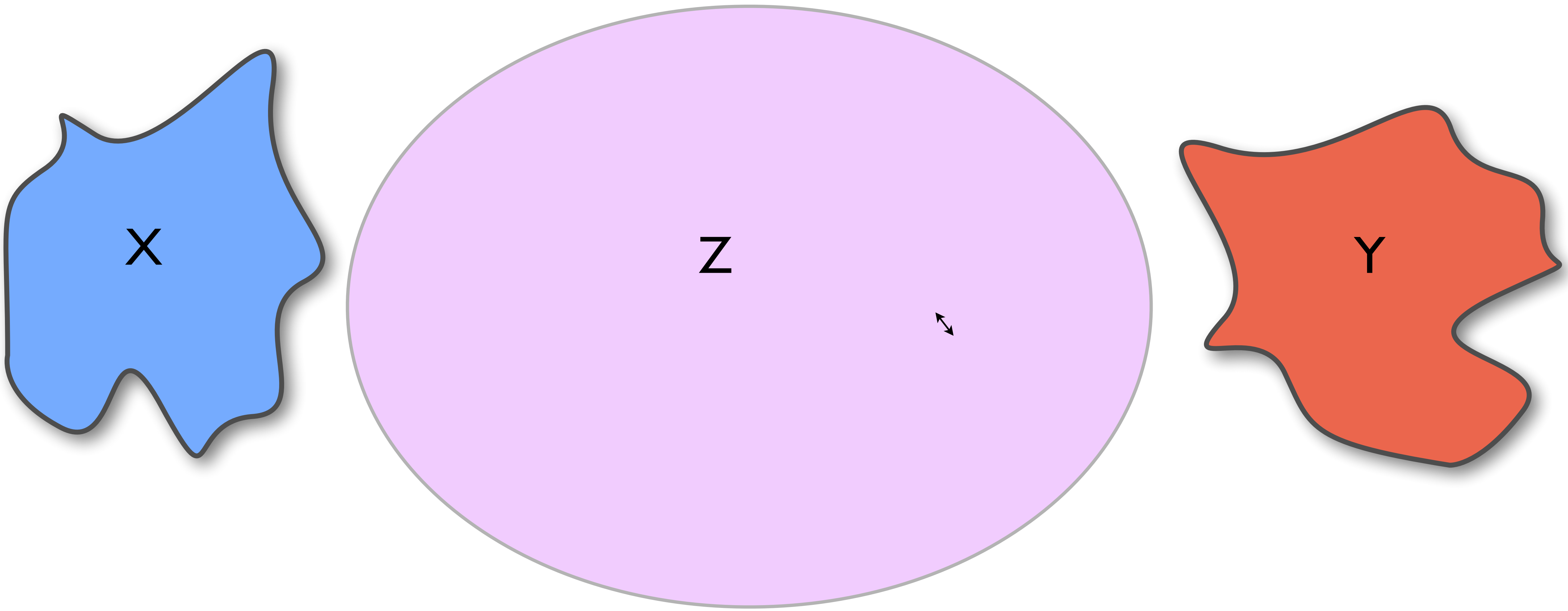

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y in a *same* metric space Z.

The Brownian tree

⇝ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

$$\frac{\sigma}{2\sqrt{n}} \cdot t_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_{\mathbb{e}},$$

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III. Plane non-crossing configurations (2012)

IV. Random maps (2004 – ?)

Scaling limits: domain of attraction of a stable law

Fix $\alpha \in (1, 2)$. Let μ be an offspring distribution such that

$$\sum_{i \geqslant 0} i\mu(i) = 1 \qquad (\mu \text{ is critical})$$

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What does t_n look like for large n?

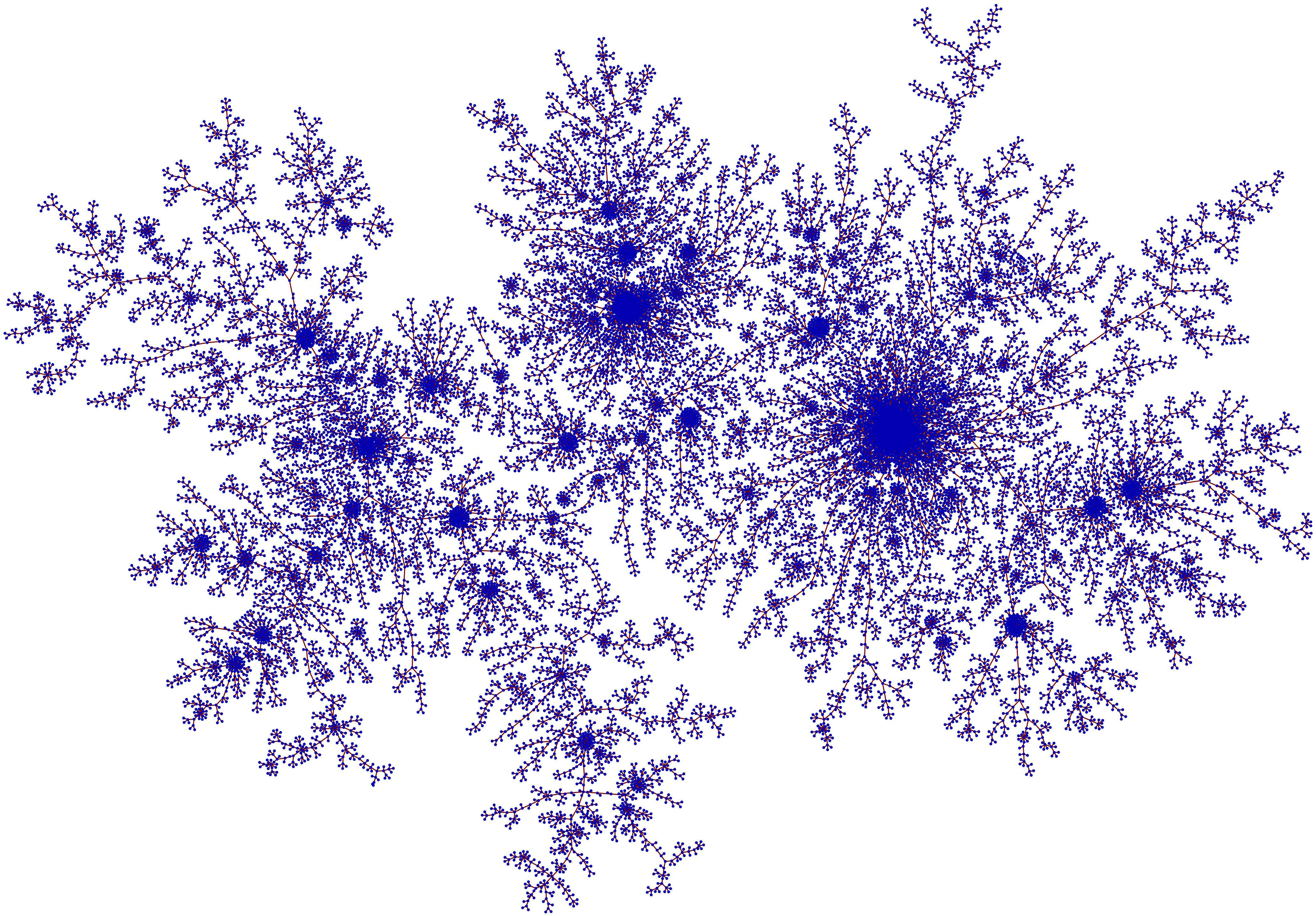

Figure: A large $\alpha = 1.1$ – stable tree

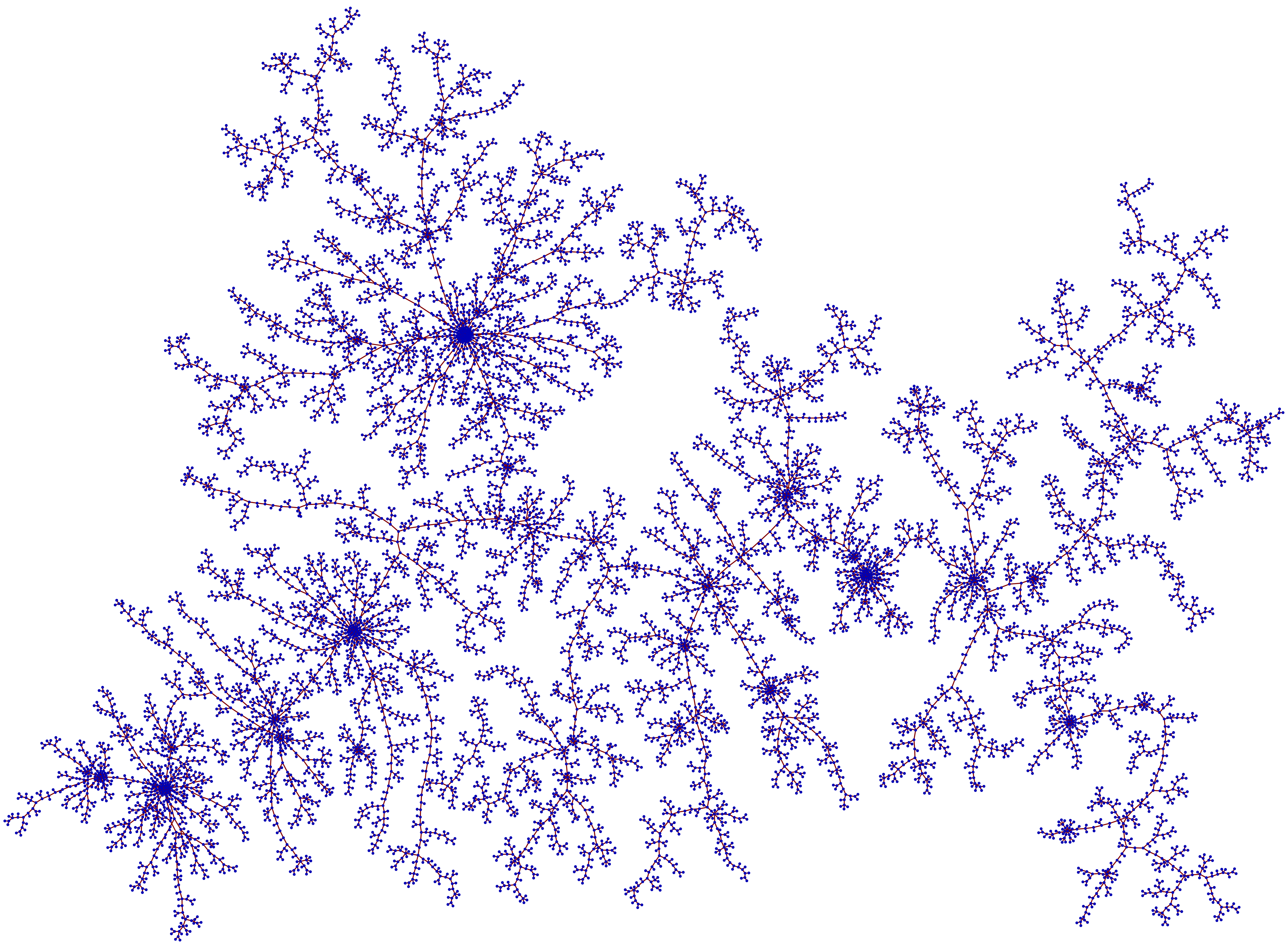

Figure: A large $\alpha = 1.5$ – stable tree

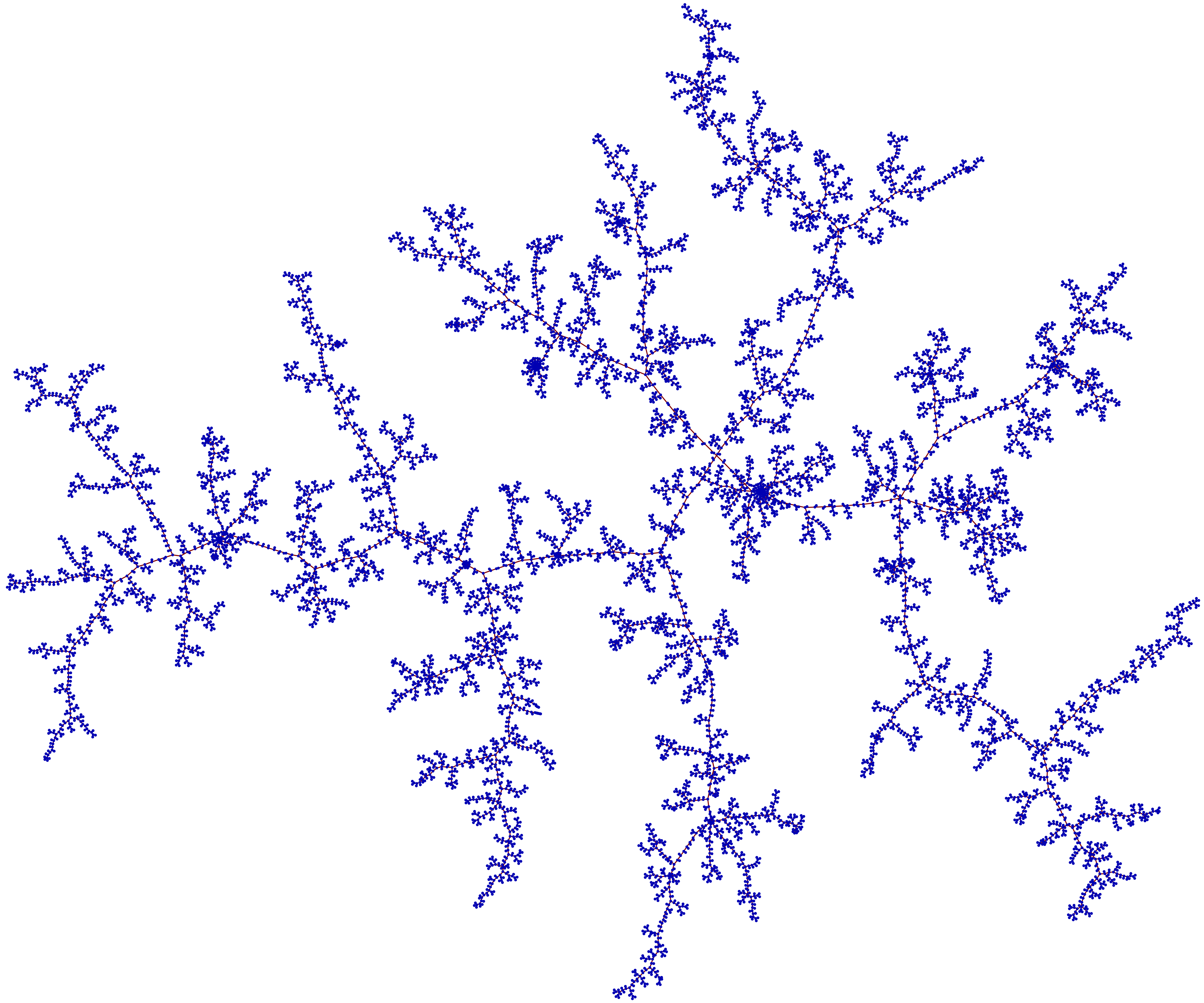

Figure: A large $\alpha = 1.9$ – stable tree

Convergence of the contour function

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- The Lukasiewicz path of $\mathcal{T}_n$, appropriately scaled, converges in distribution to the normalized excursion of a spectrally positive stable Lévy process of index α (conditioned Donsker's invariance principle).

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Simulations of $\mathcal{H}^{(\alpha)}$

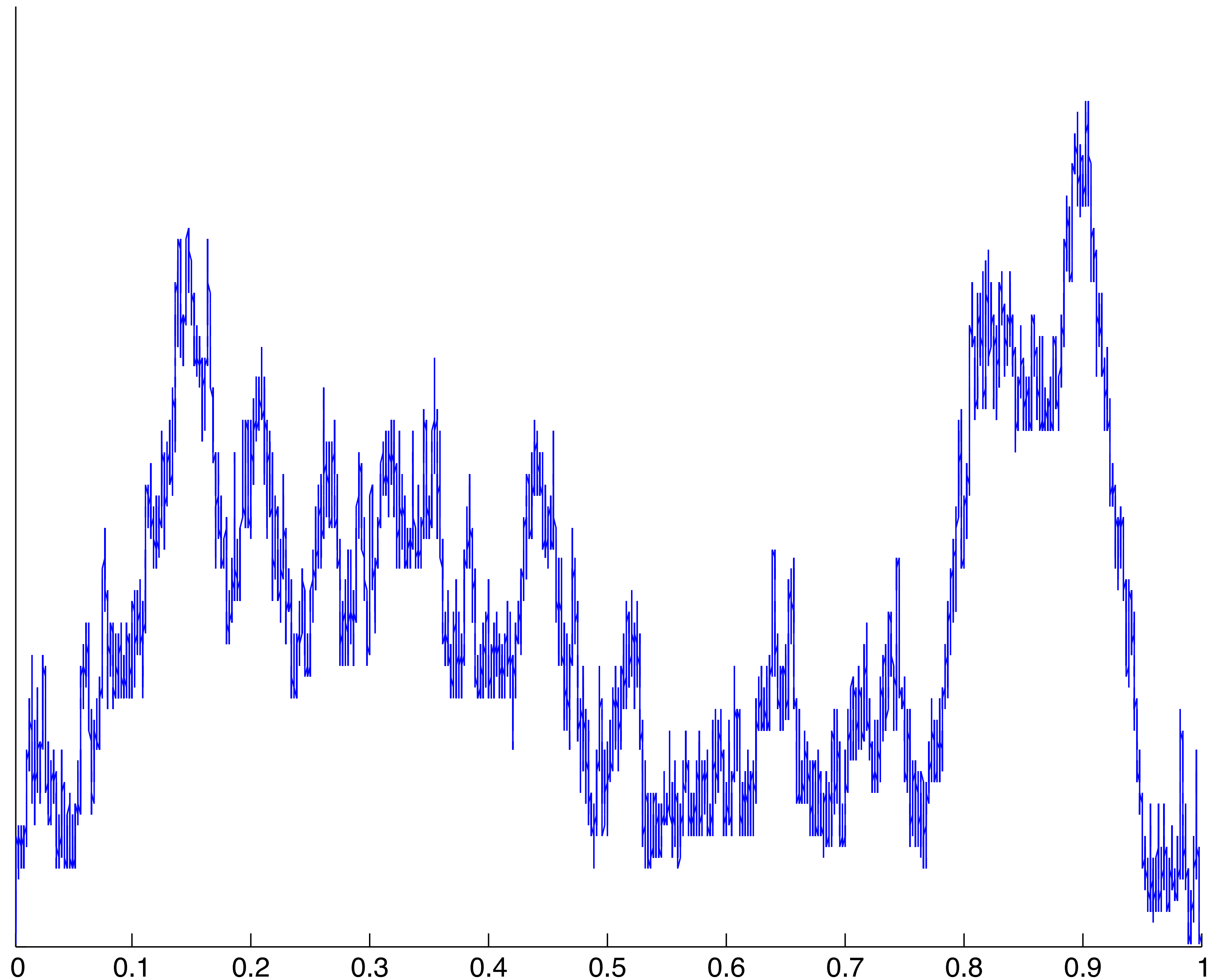

Figure: Simulation of $\mathcal{H}^{\alpha}$ for $\alpha = 1.1$.

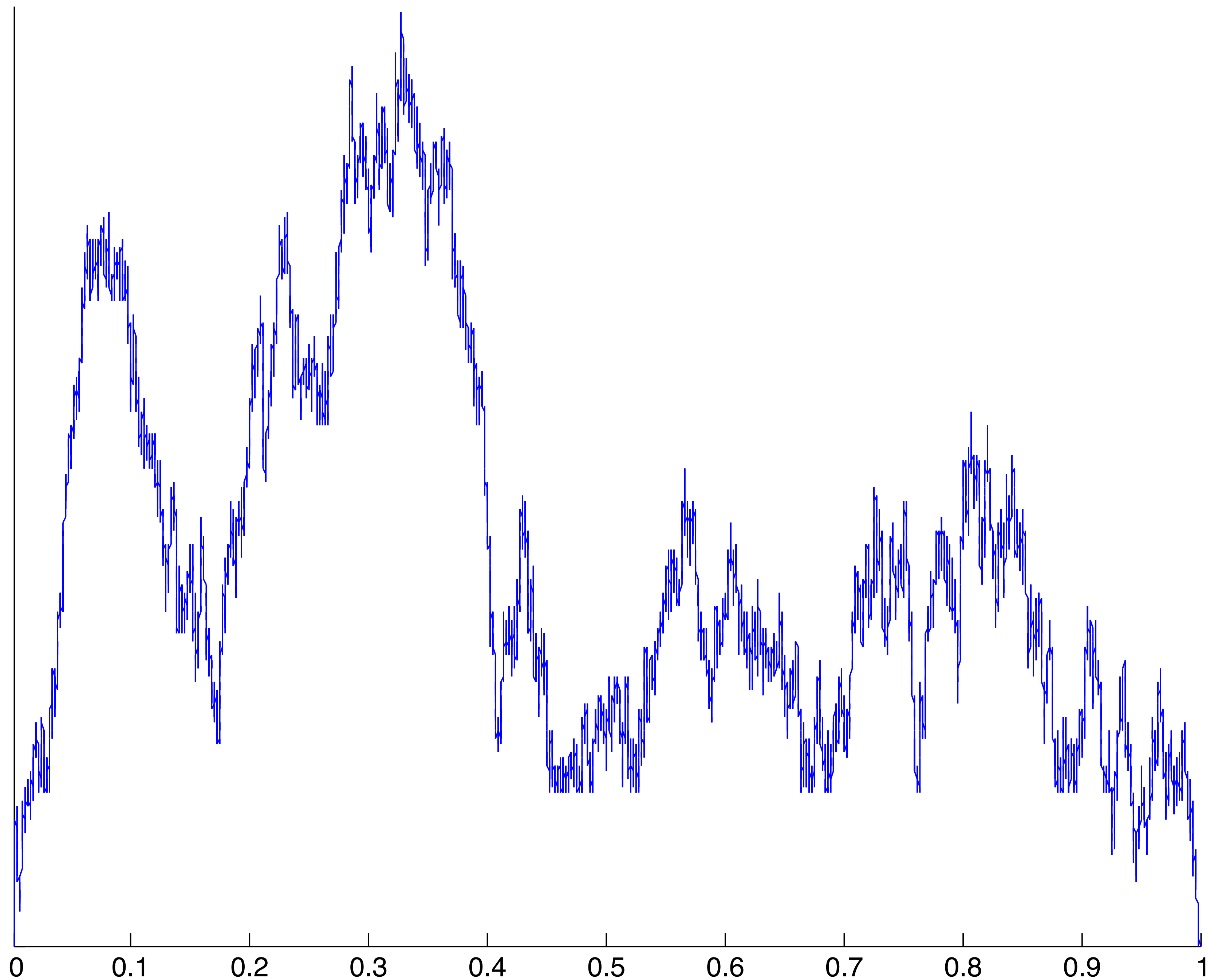

Figure: Simulation of $\mathcal{H}^{\alpha}$ for $\alpha = 1.6$.

Scaling limits in the Gromov–Hausdorff topology

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⇝ $\mathcal{T}_{\alpha}$ is coded by $\mathcal{H}^{(\alpha)}$.

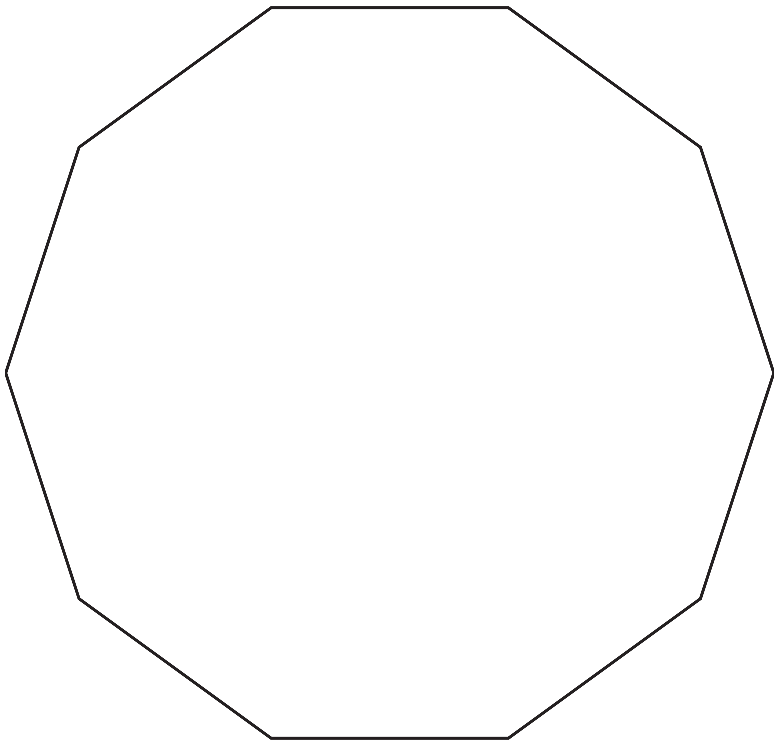
I. Scaling limits of BGW trees (finite variance, 1991)

II. Scaling limits of BGW trees (infinite variance, 1998)

III. Plane non-crossing configurations (2012)

IV. Random maps (2004 – ?)

Let P_n be the polygon with vertices $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$.



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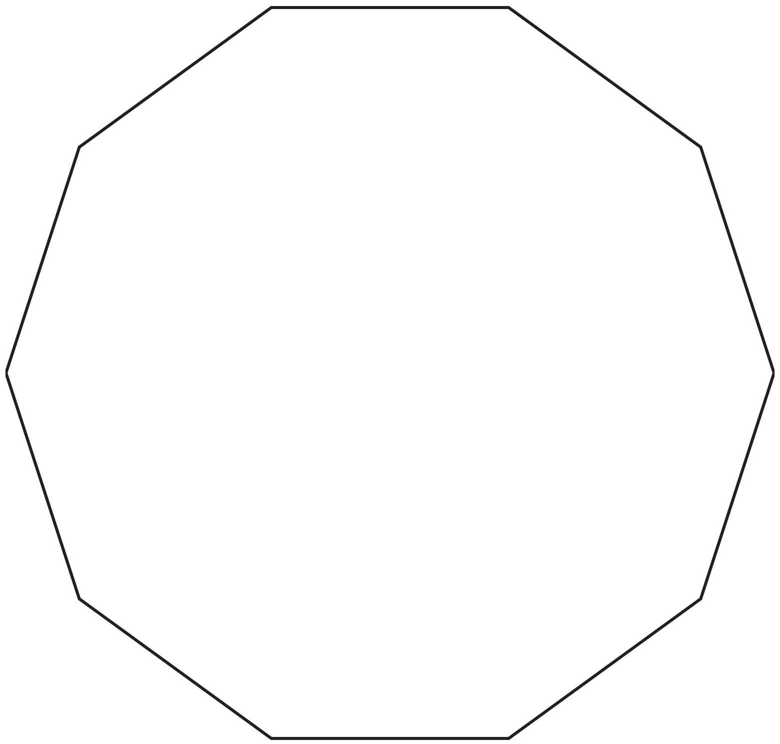

General philosophy: chose at random a non crossing configuration, obtained from the vertices of P_n by drawing diagonals which may not cross.

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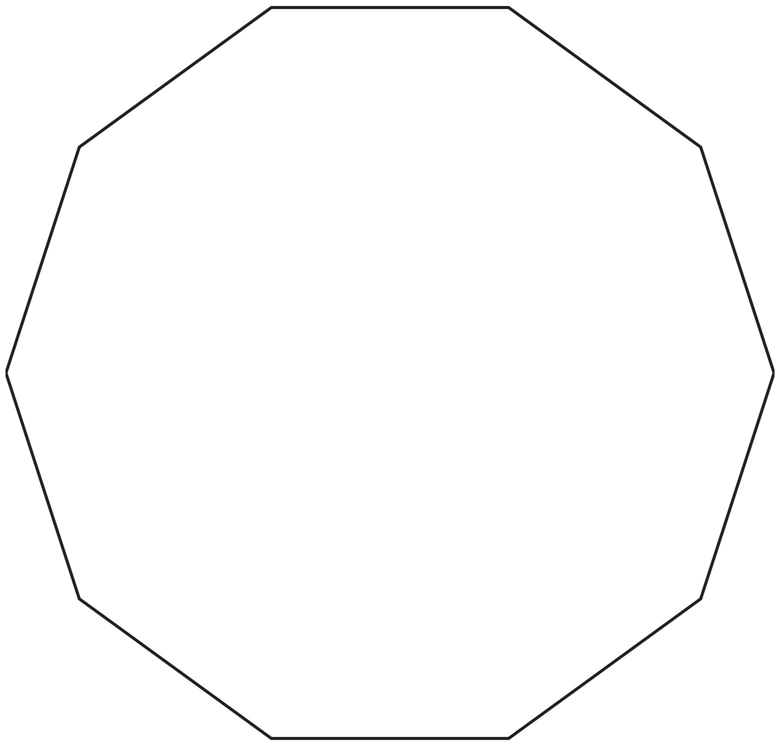

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What happens for n large?

Case of triangulations of P_n

Let $\mathcal{T}_n$ be a random triangulation, chosen uniformly among all triangulations of P_n. What does $\mathcal{T}_n$ look like when n is large?

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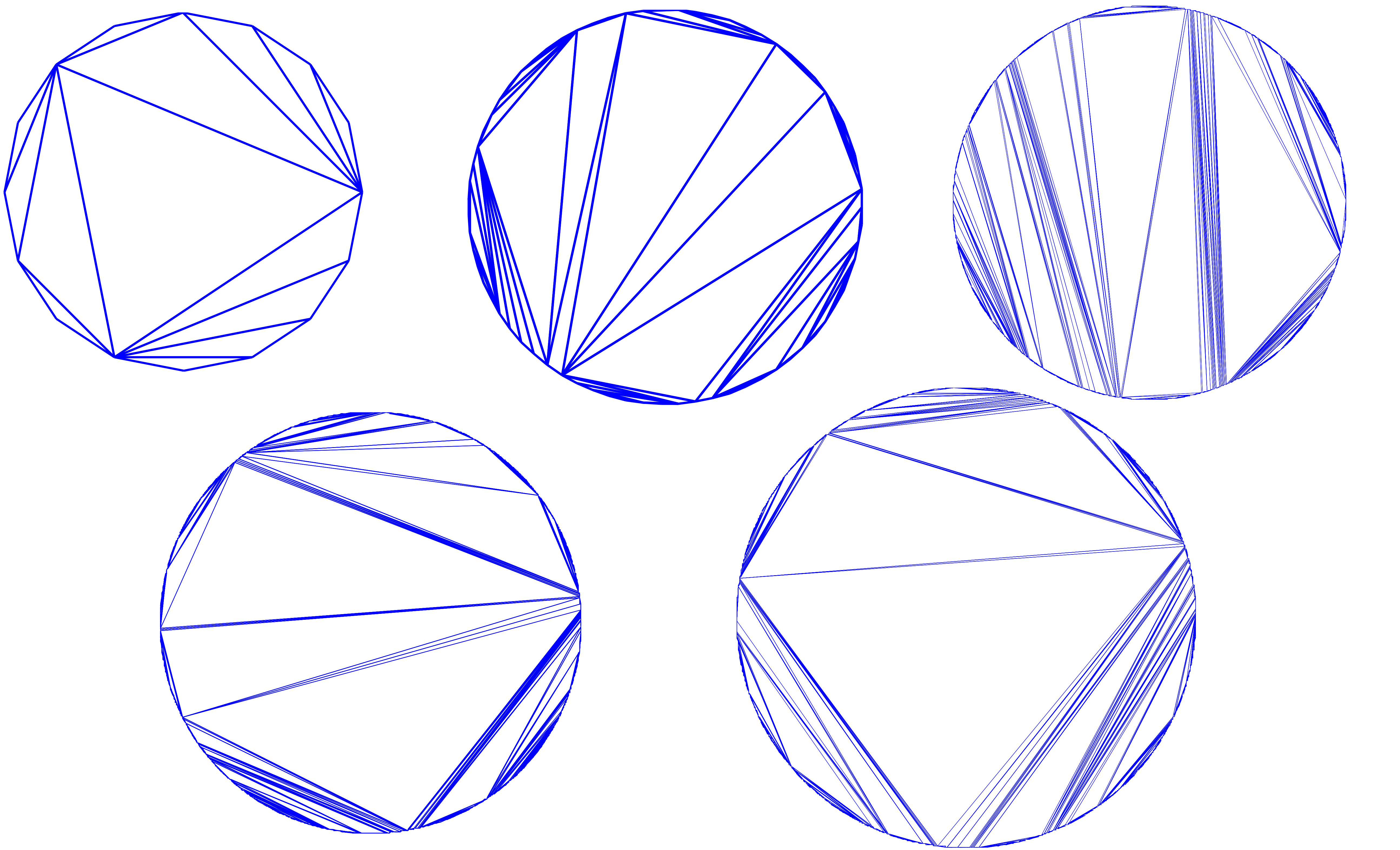

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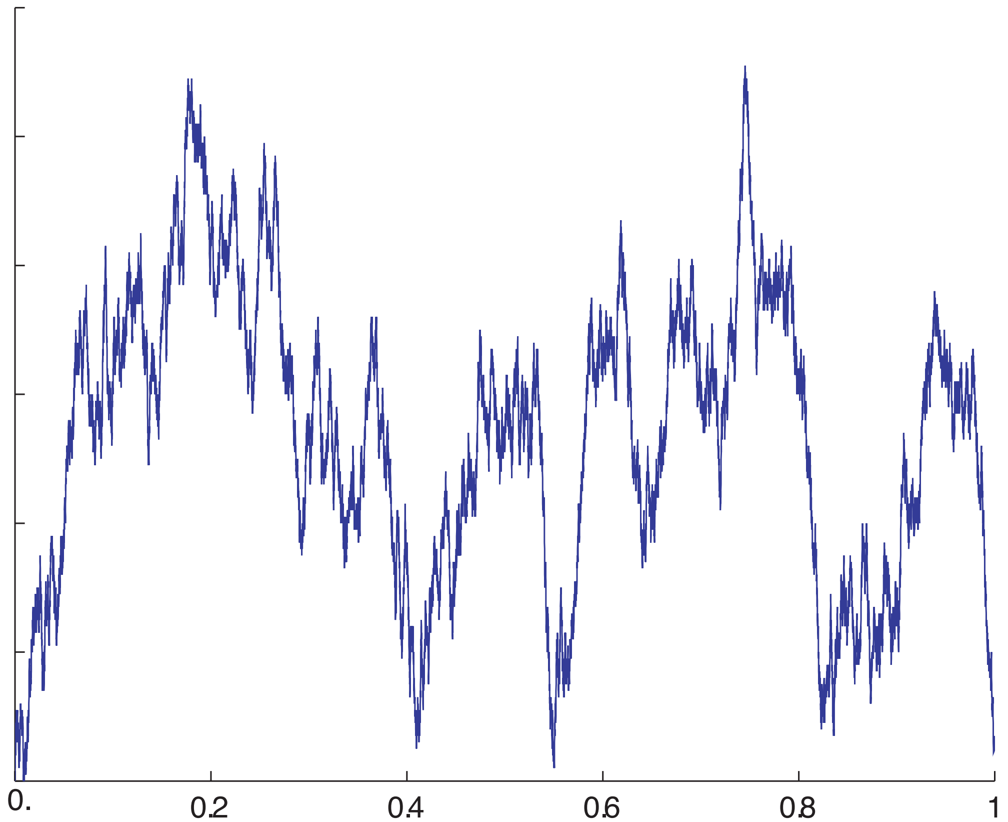

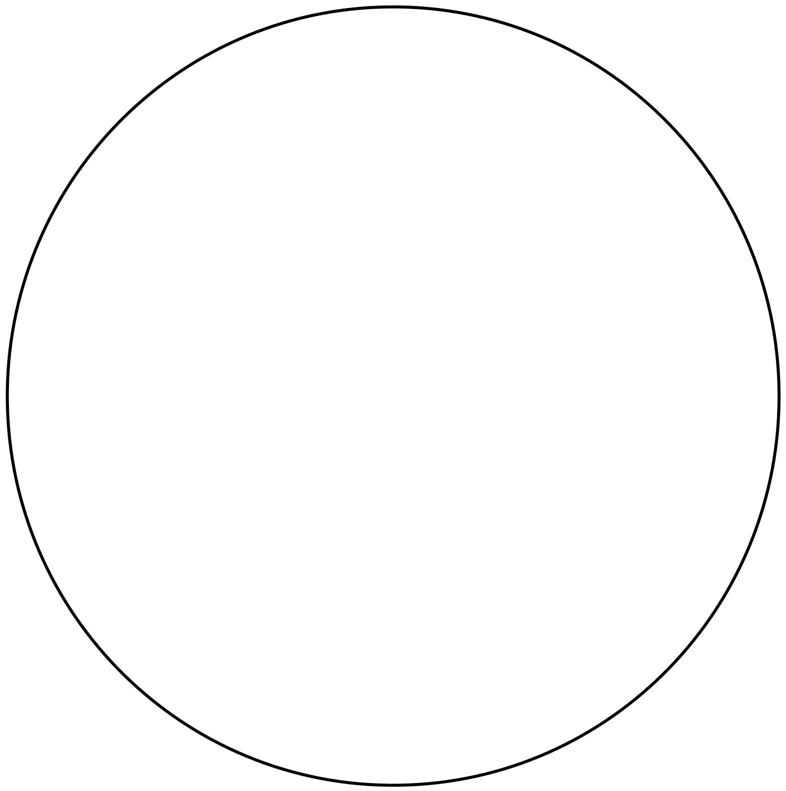

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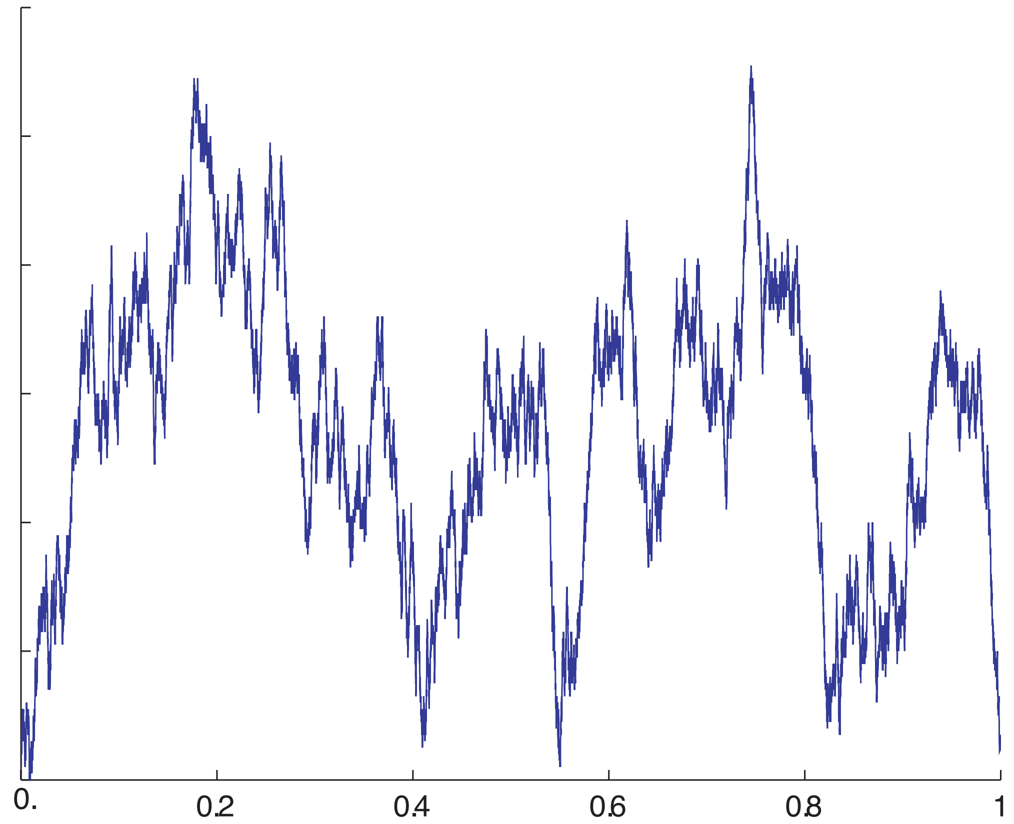

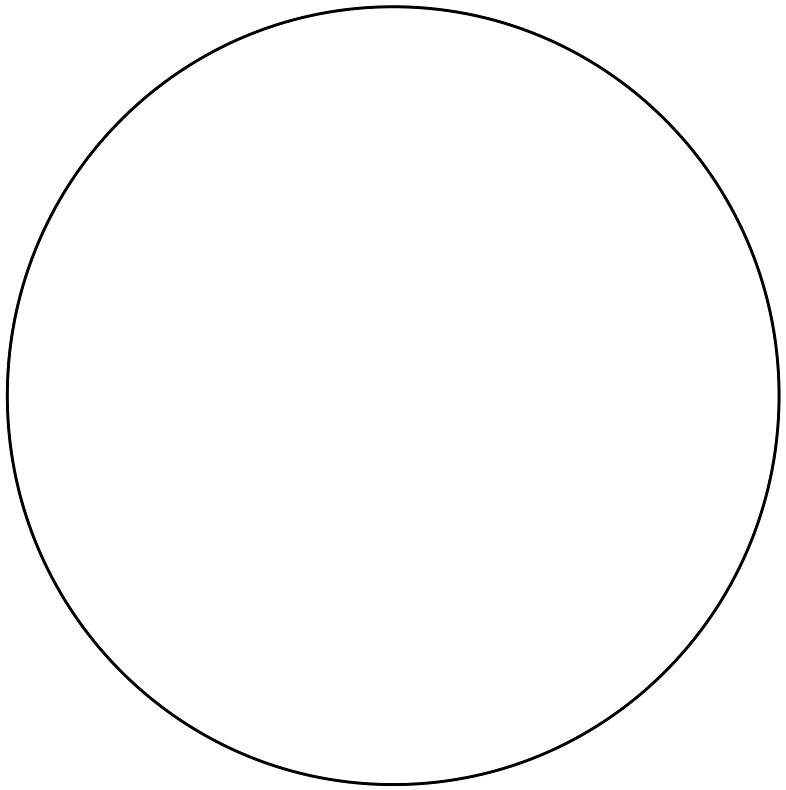

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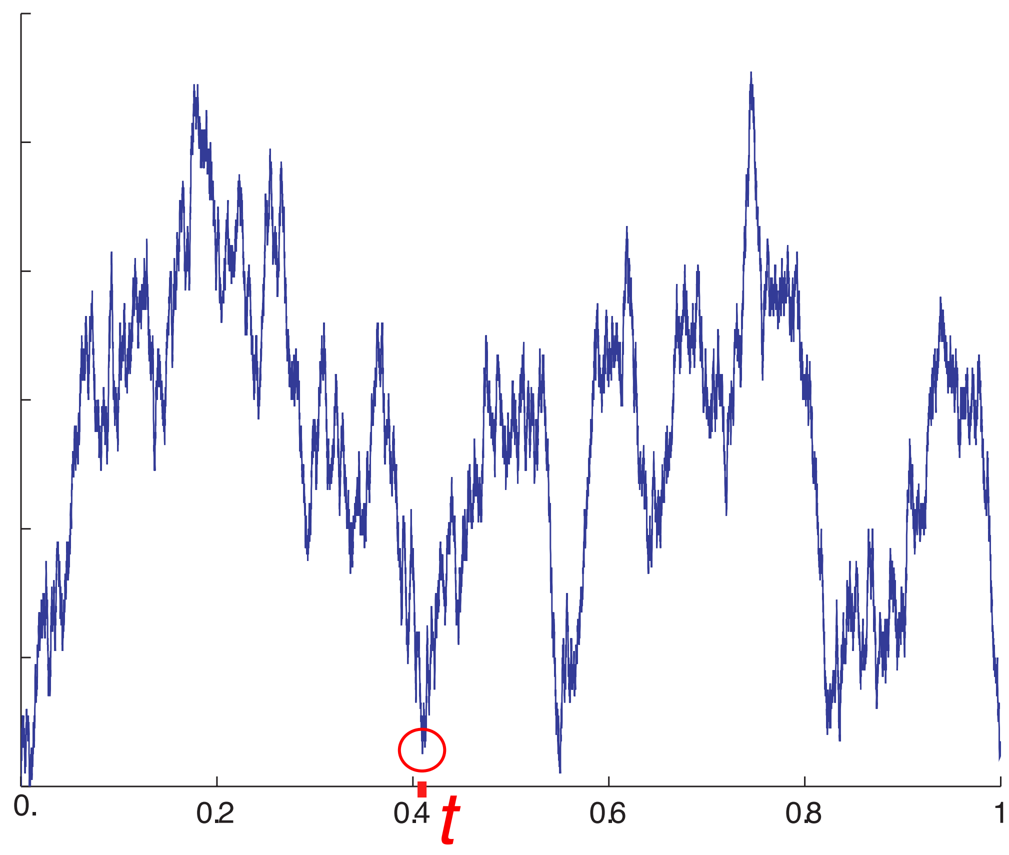

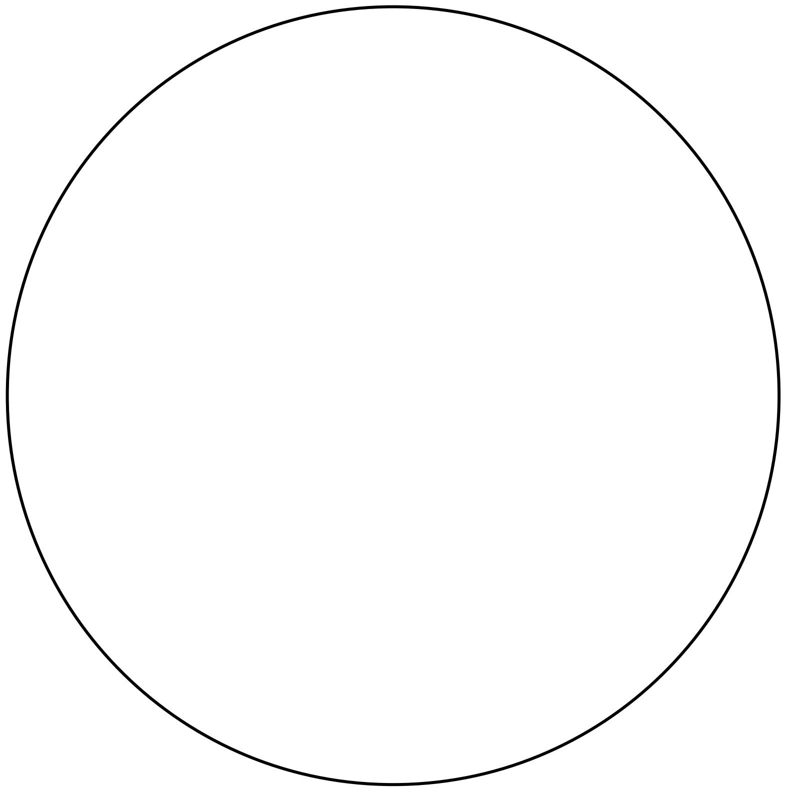

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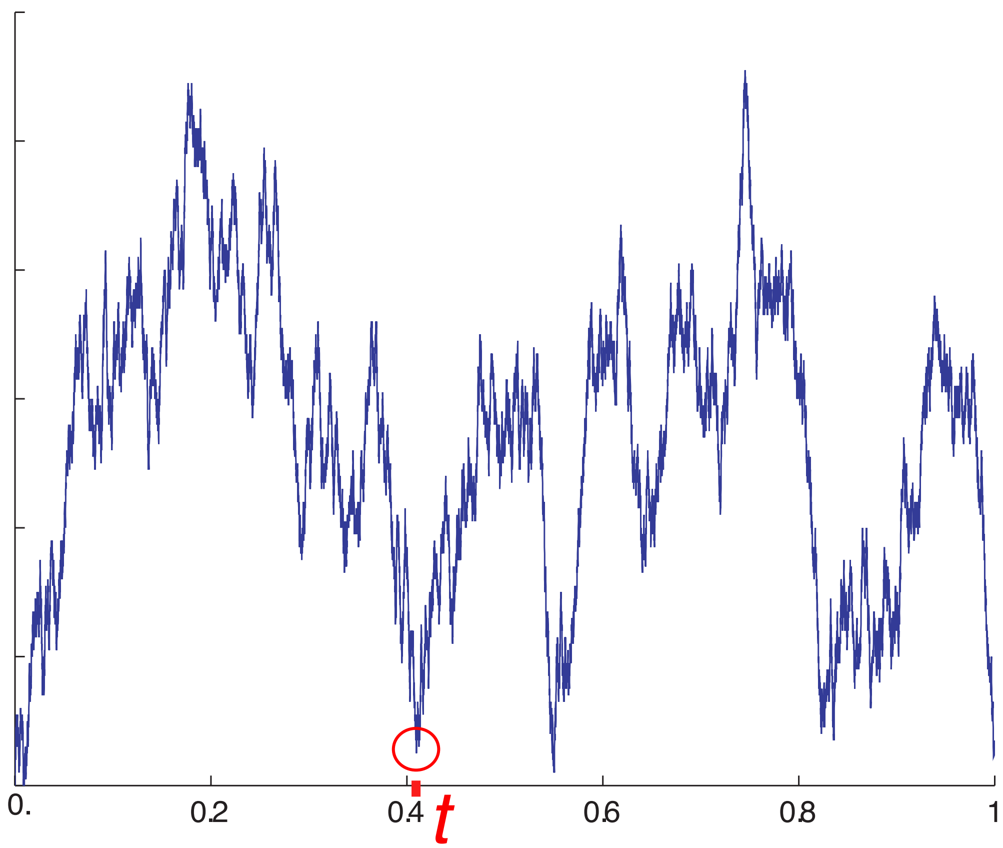

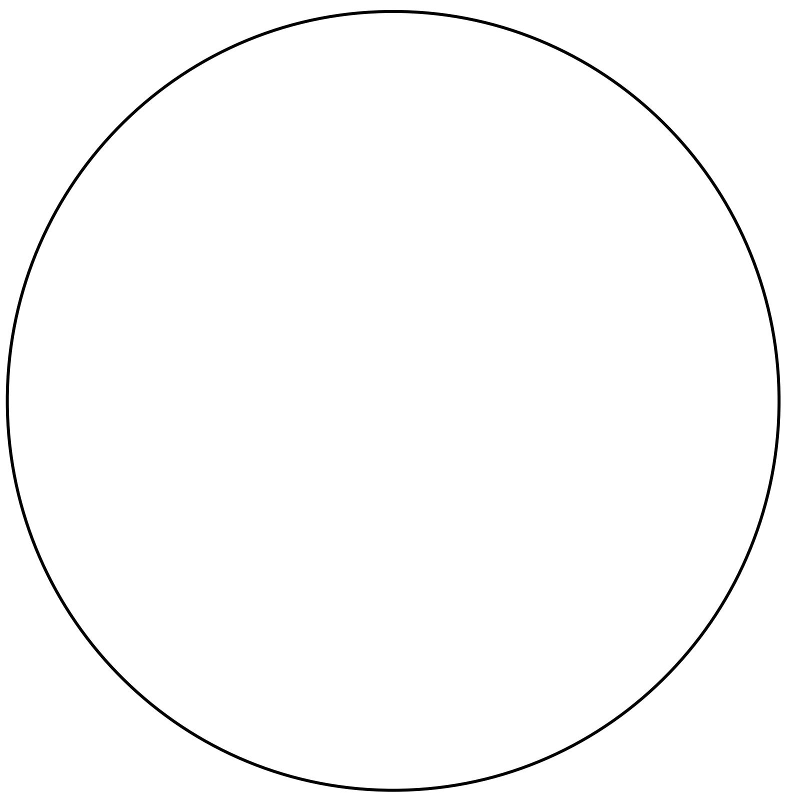

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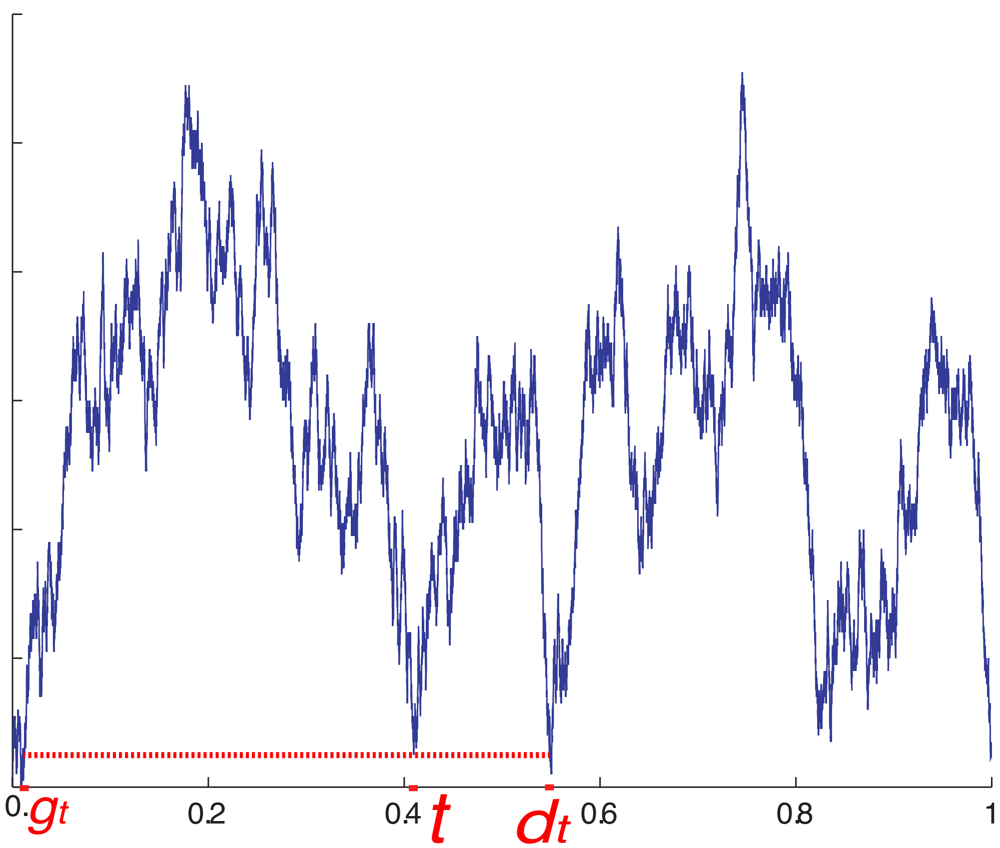

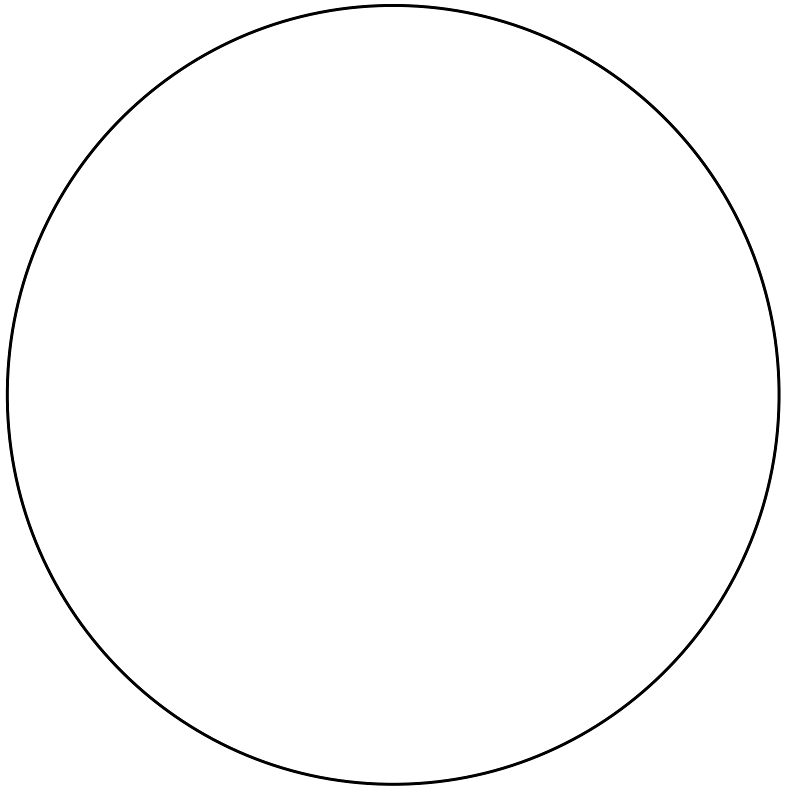

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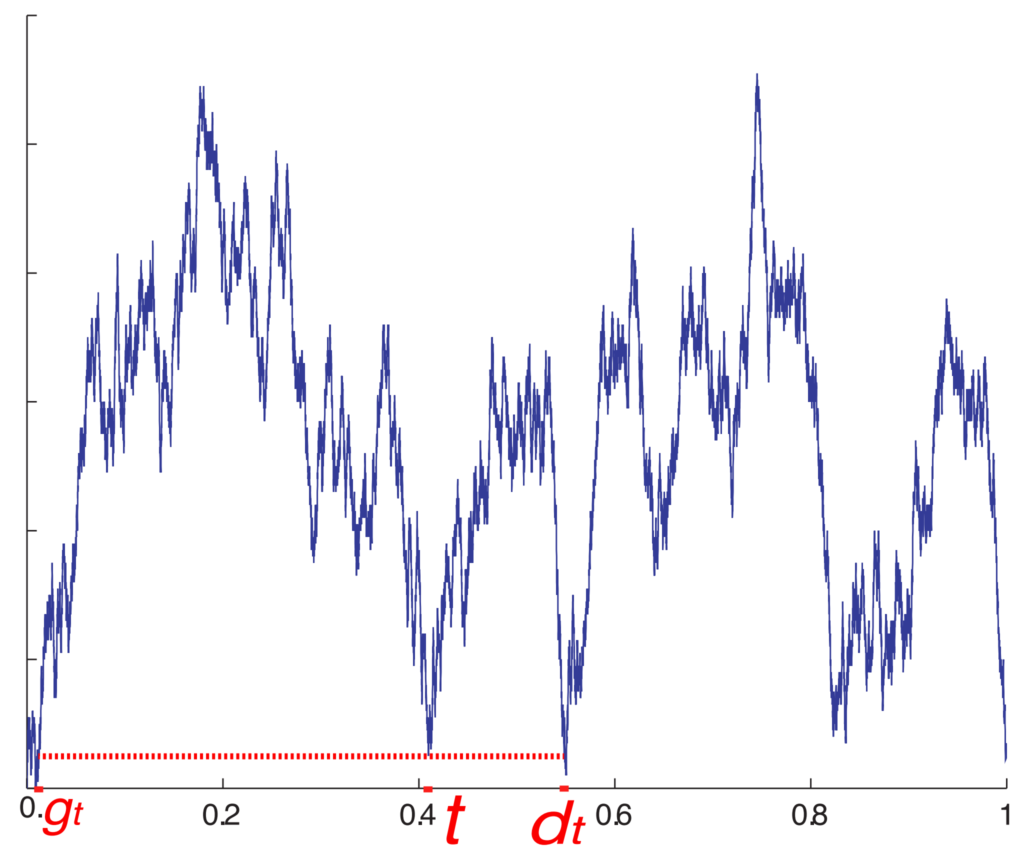

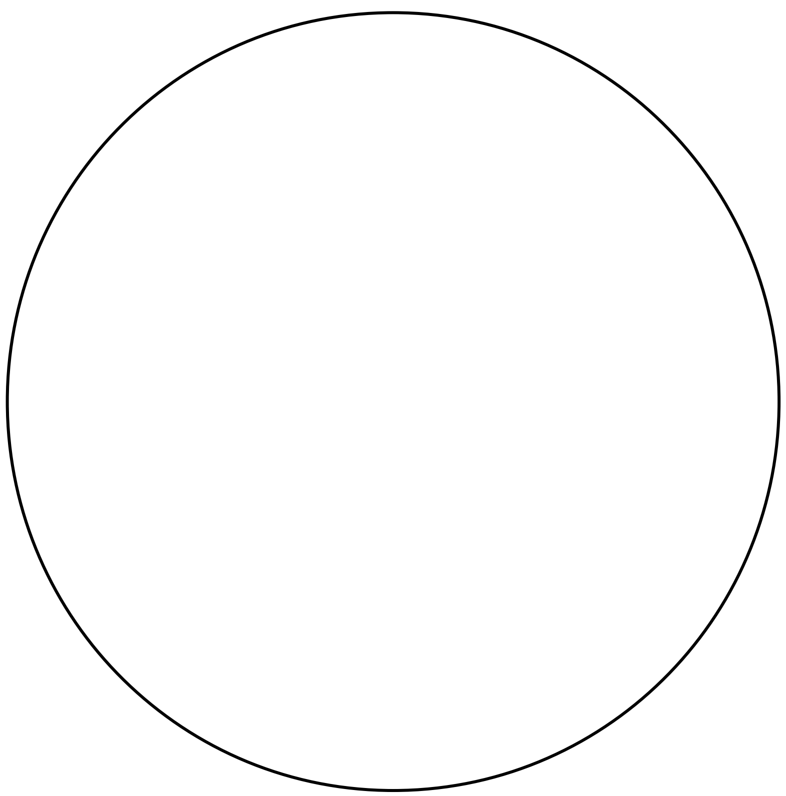

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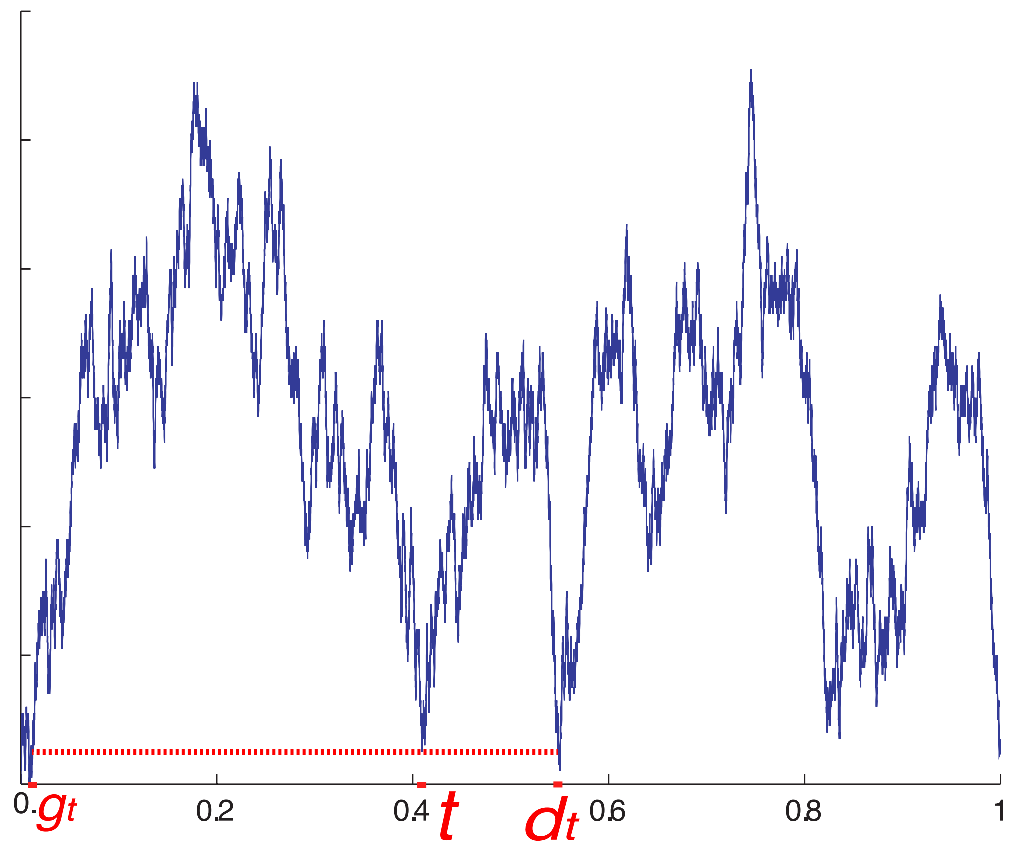

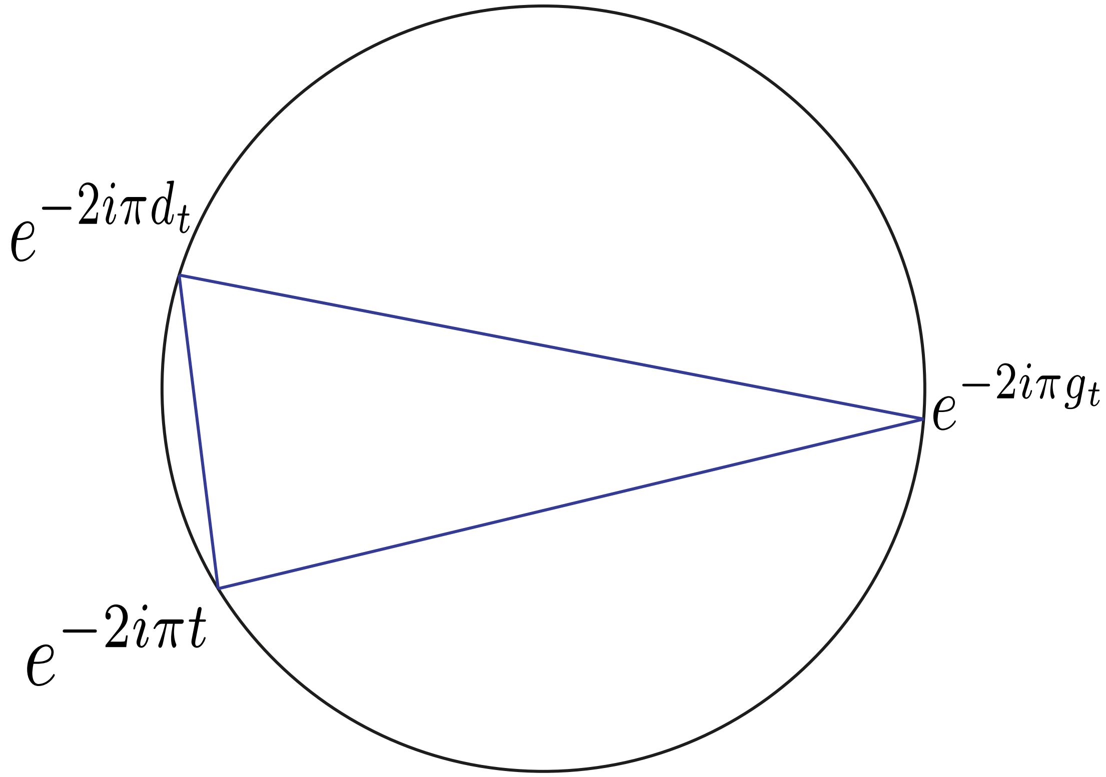

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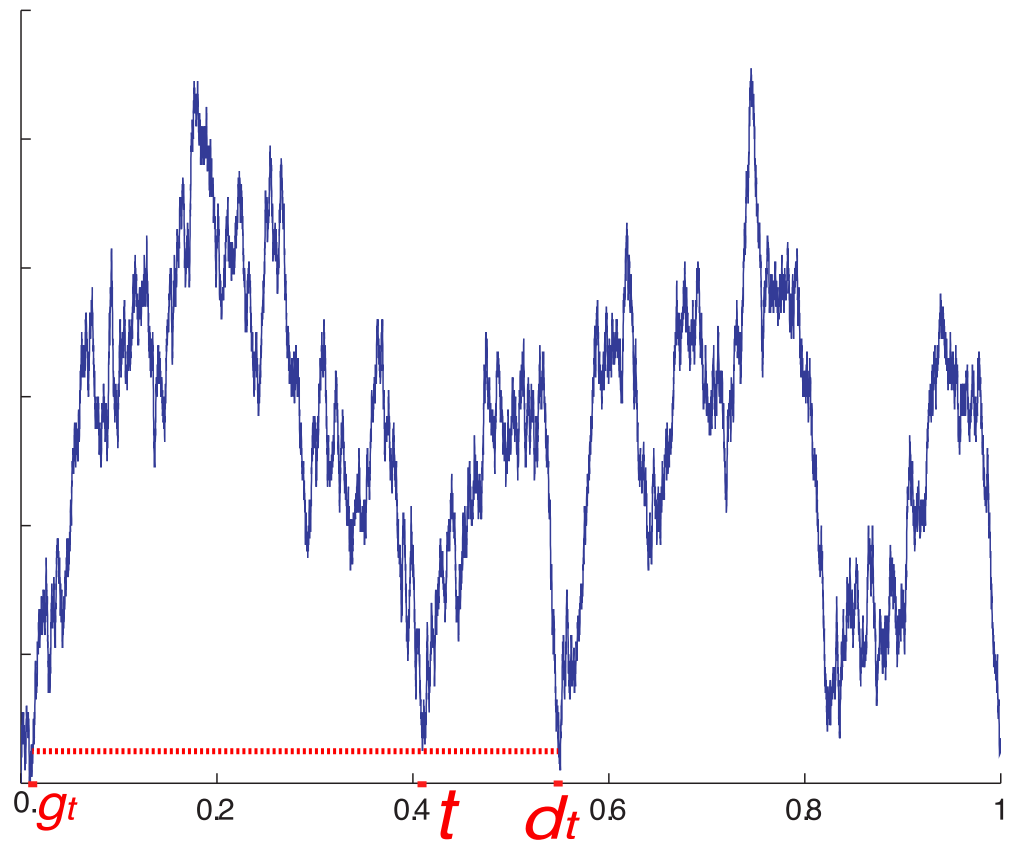

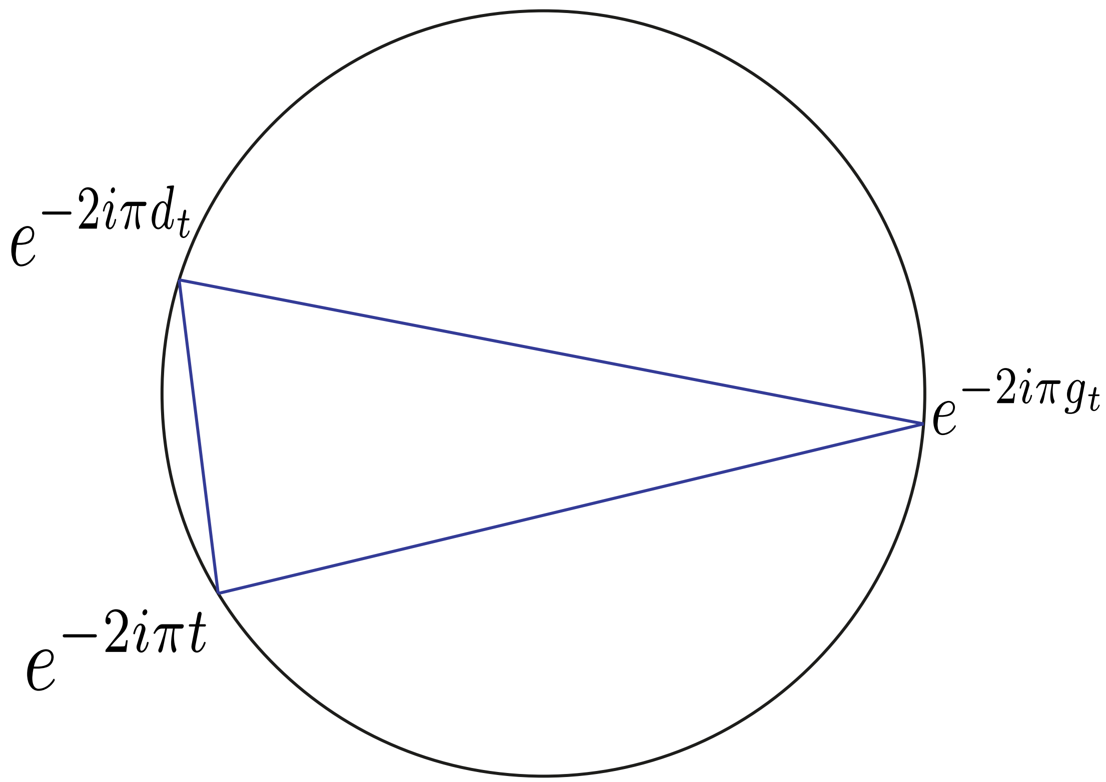

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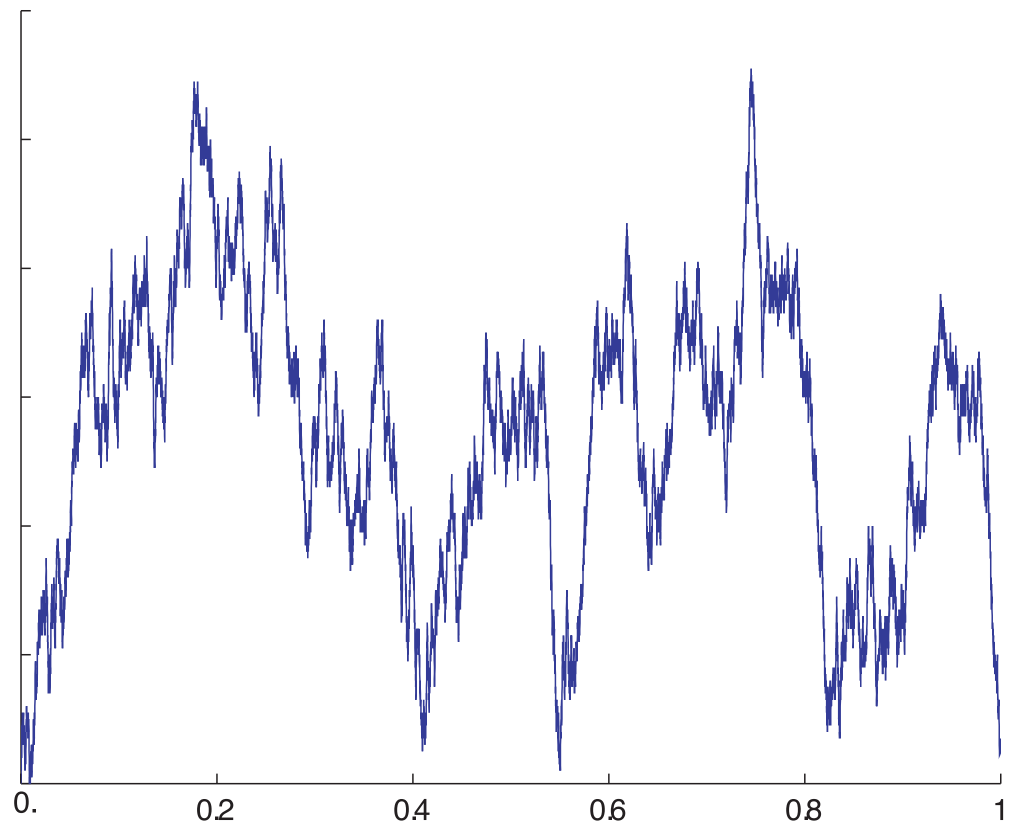

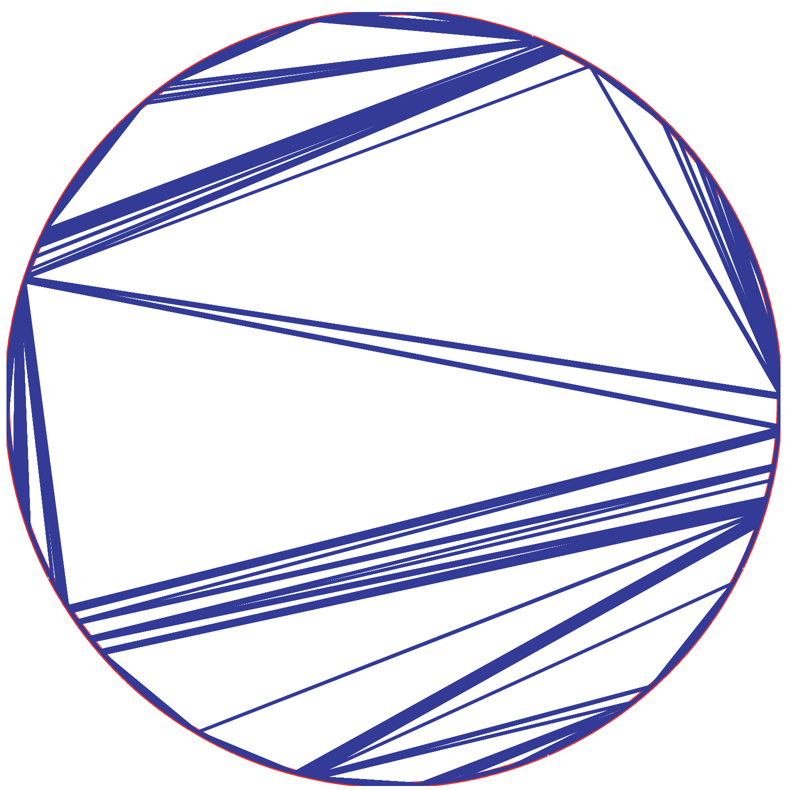

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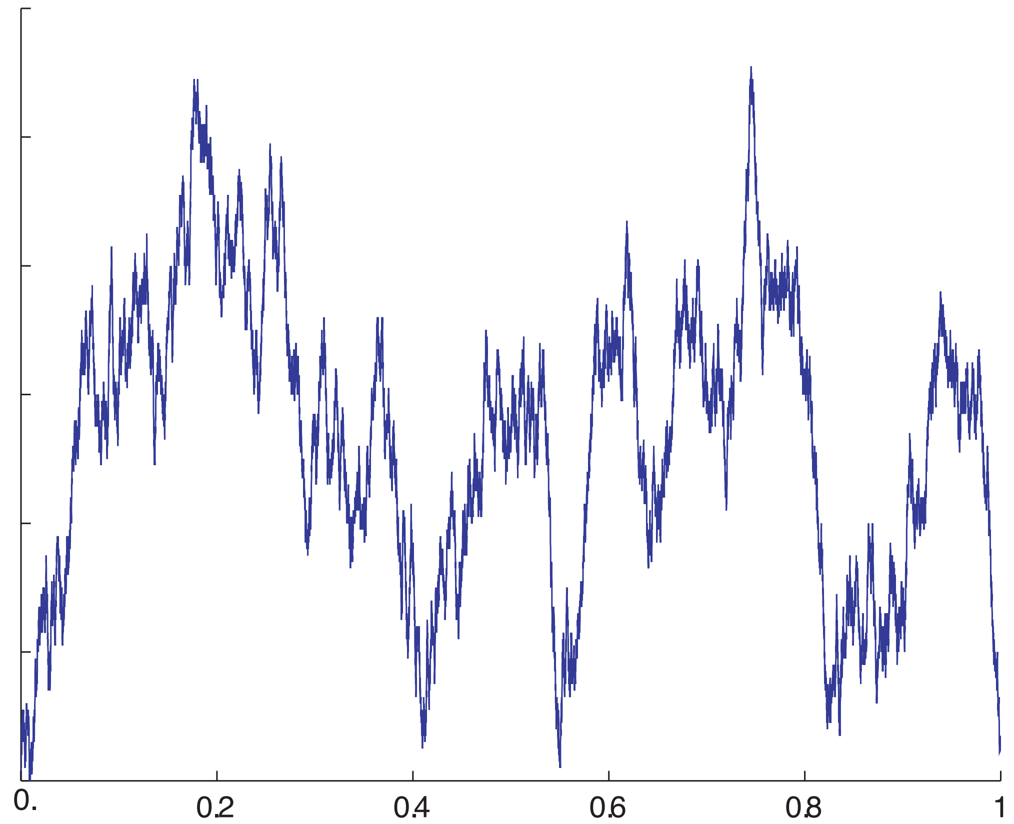

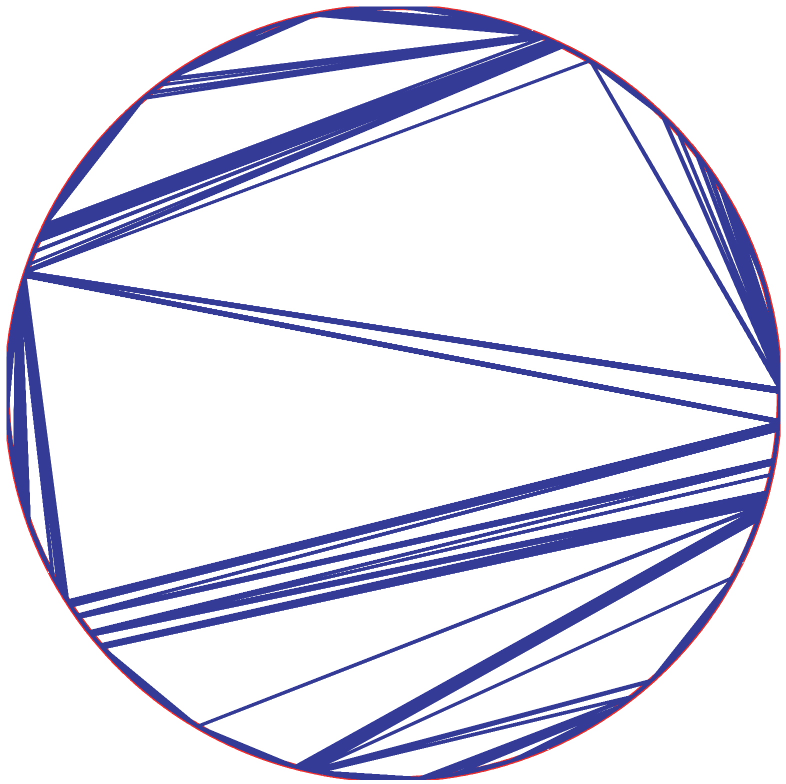

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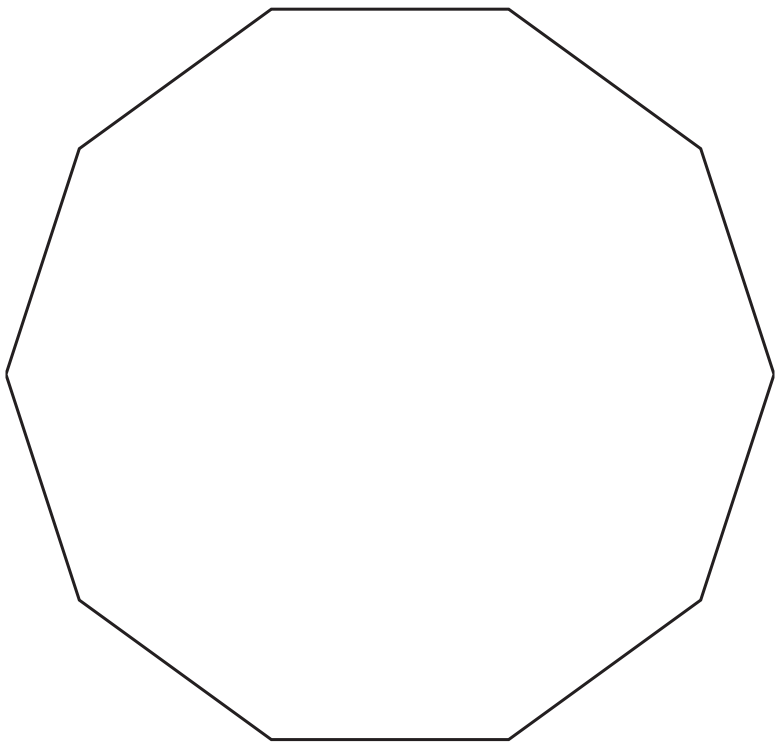
Do this for all local minimum times.

The closure of this object, denoted by $L(\mathbb{e})$, is called the Brownian triangulation.

Case of dissections of P_n

Dissections

Recall that P_n is the polygon with vertices $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$.



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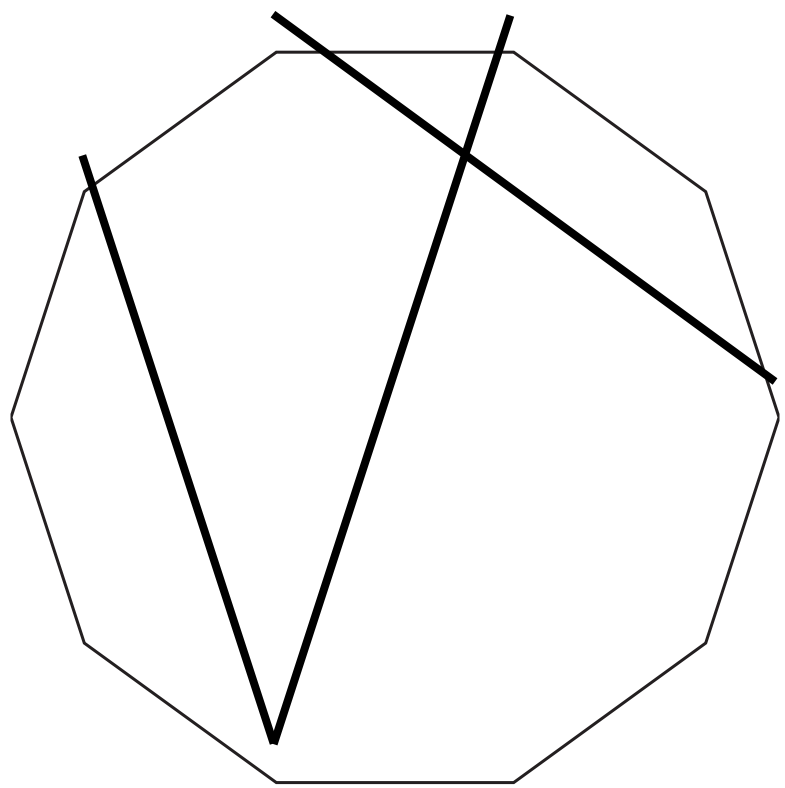

Not a dissection !

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Let $\mathcal{D}_n$ be a random dissection, chosen uniformly at random among all dissections of P_n. What does $\mathcal{D}_n$ look like as $n \to \infty$?

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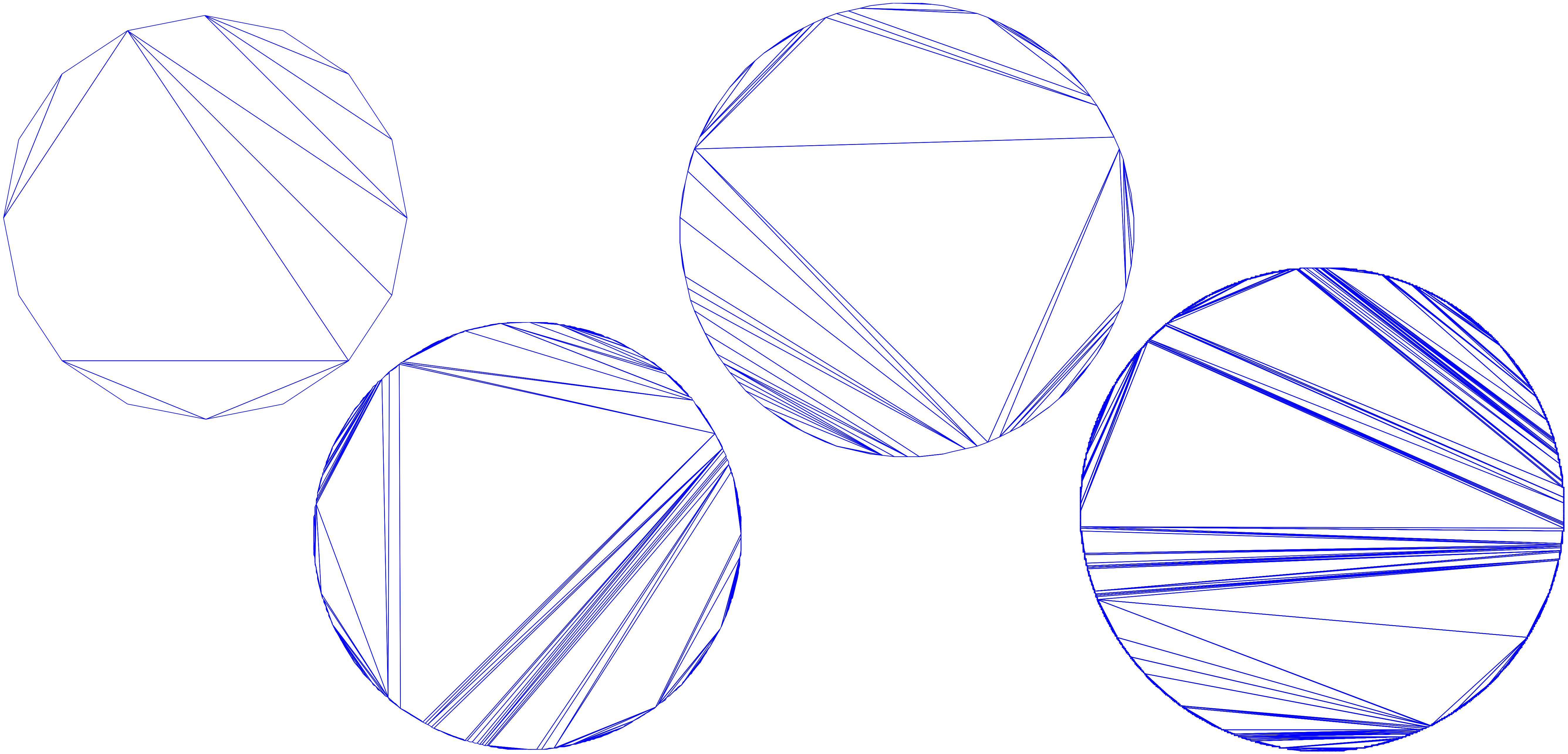

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How to prove that these models converge to the Brownian triangulation?

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Key point: these trees can be coded by BGW trees.

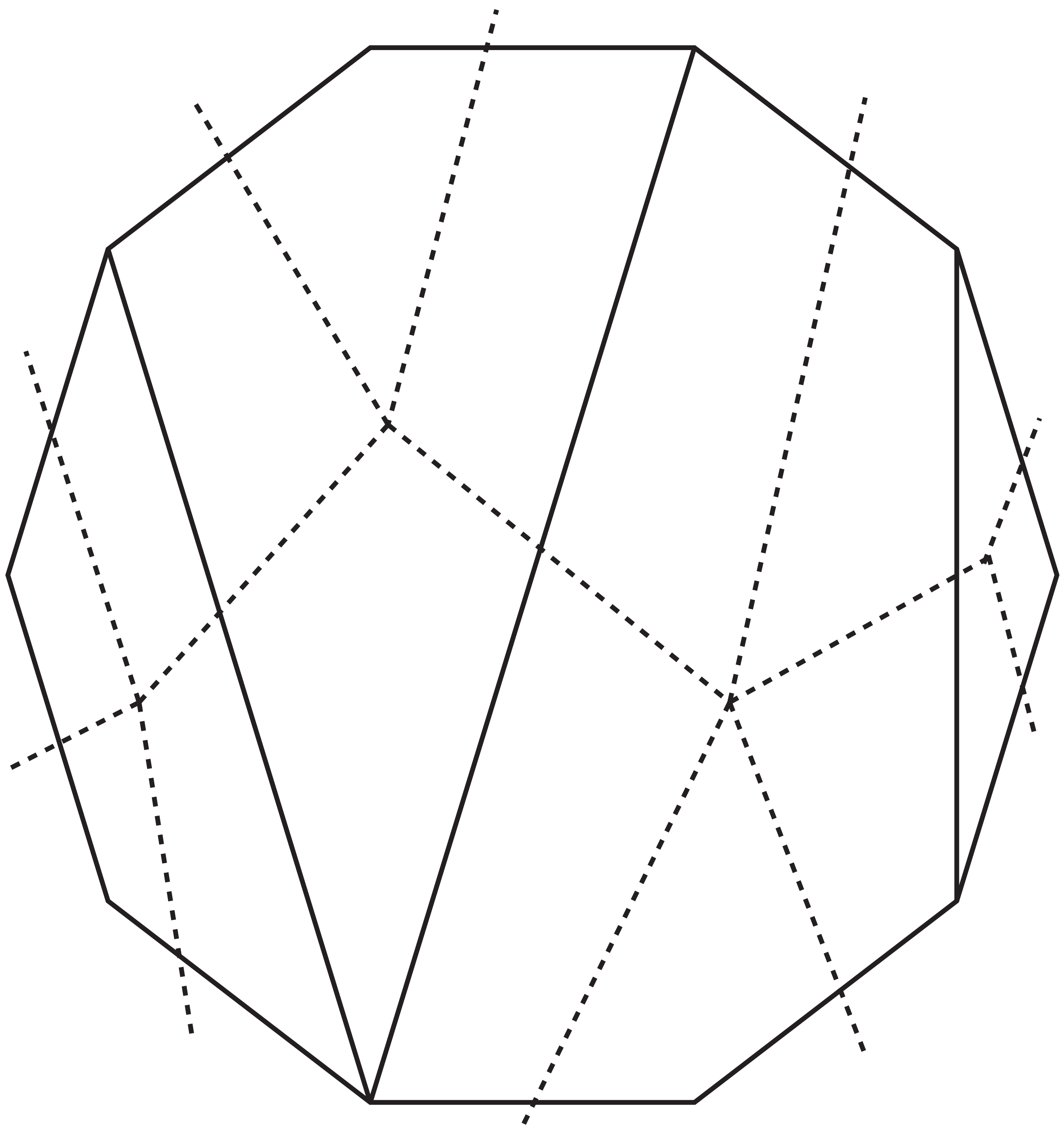


Figure: The dual tree of a dissection.

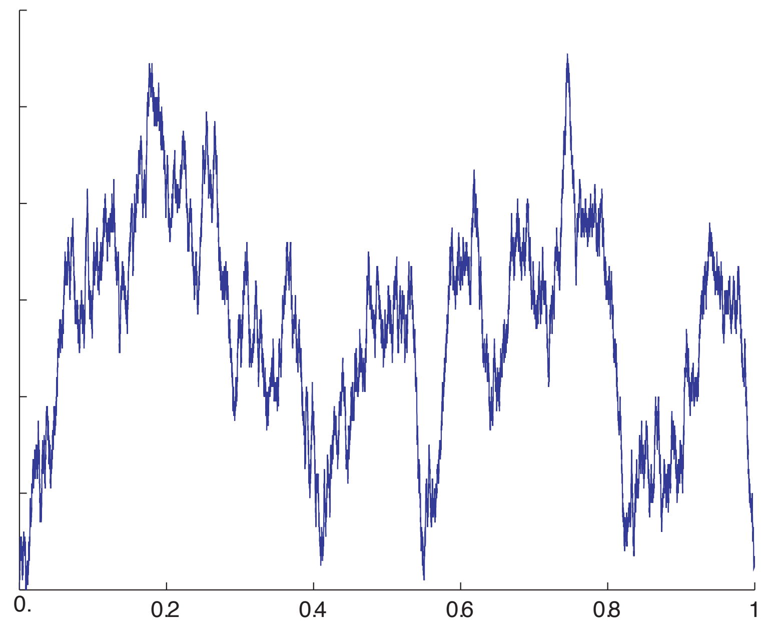
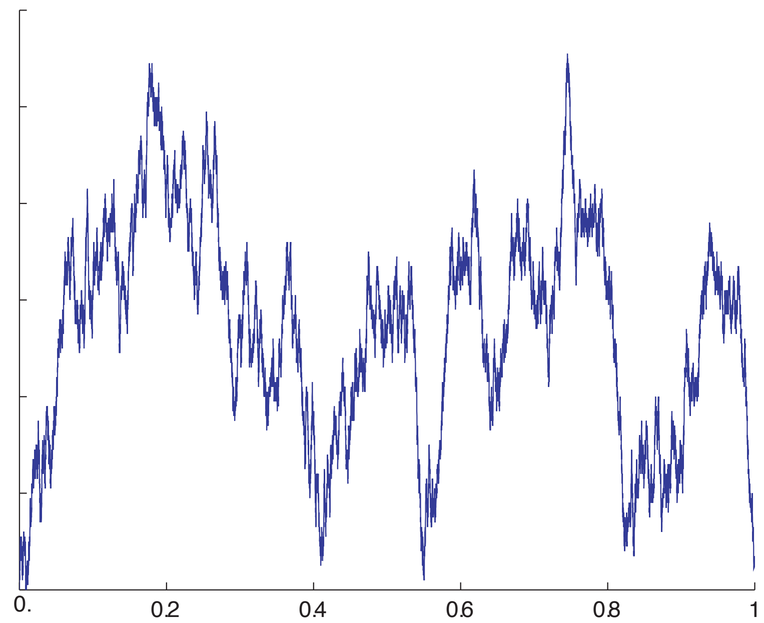


Figure: Normalized contour function of a large conditioned Bienaymé–Galton-Watson.



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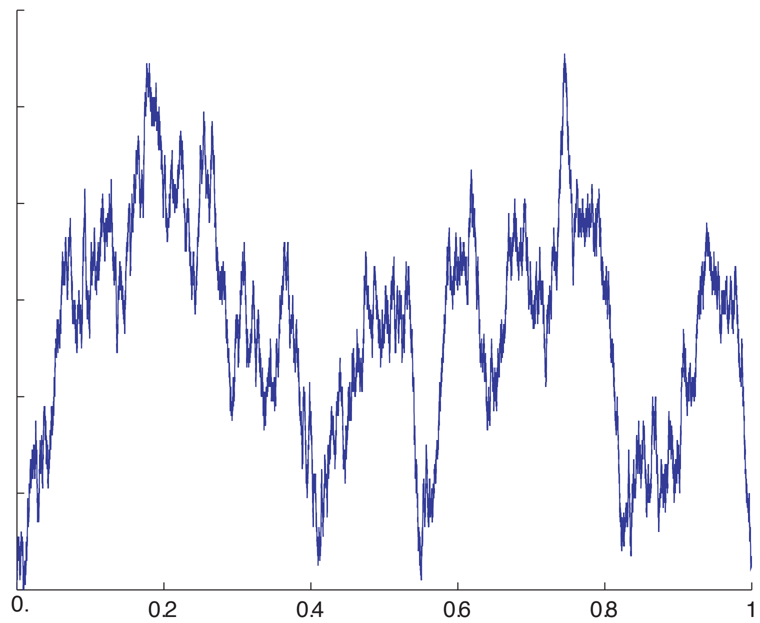

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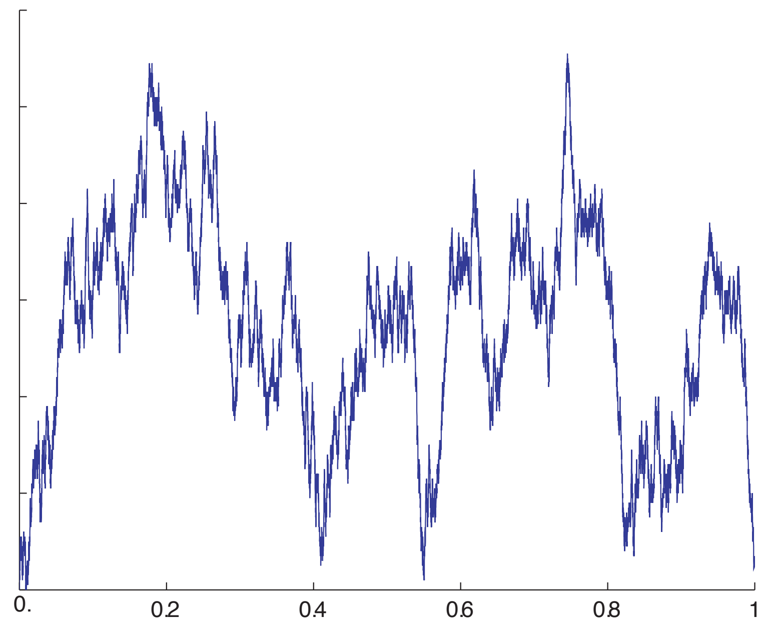

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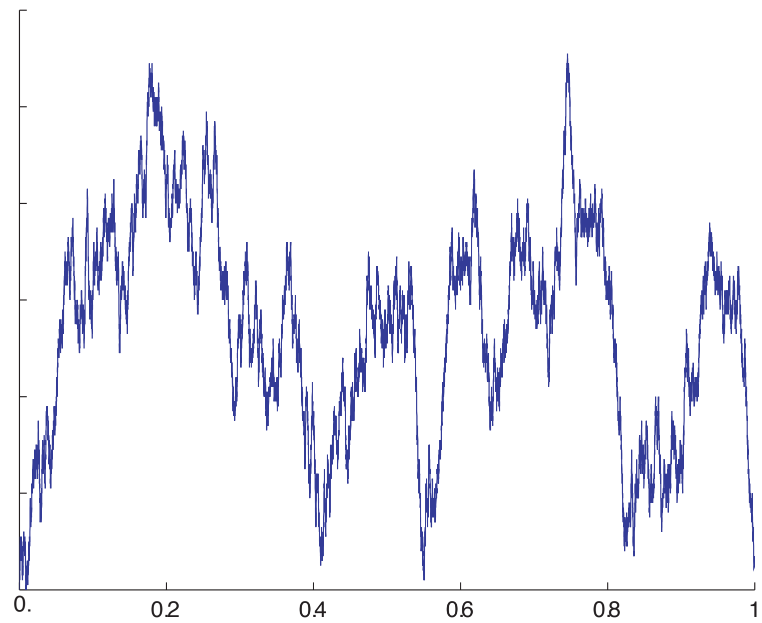

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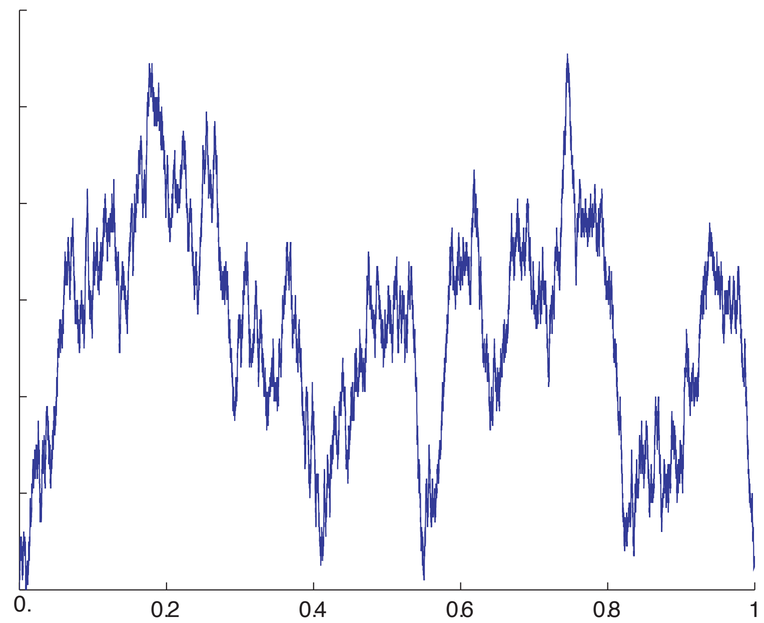

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Therefore these random plane non-crossing configurations converge to $\mathbb{L}(\mathbb{e})$.

What about dissections seen as compact metric spaces?

Dissections seen as compact metric spaces

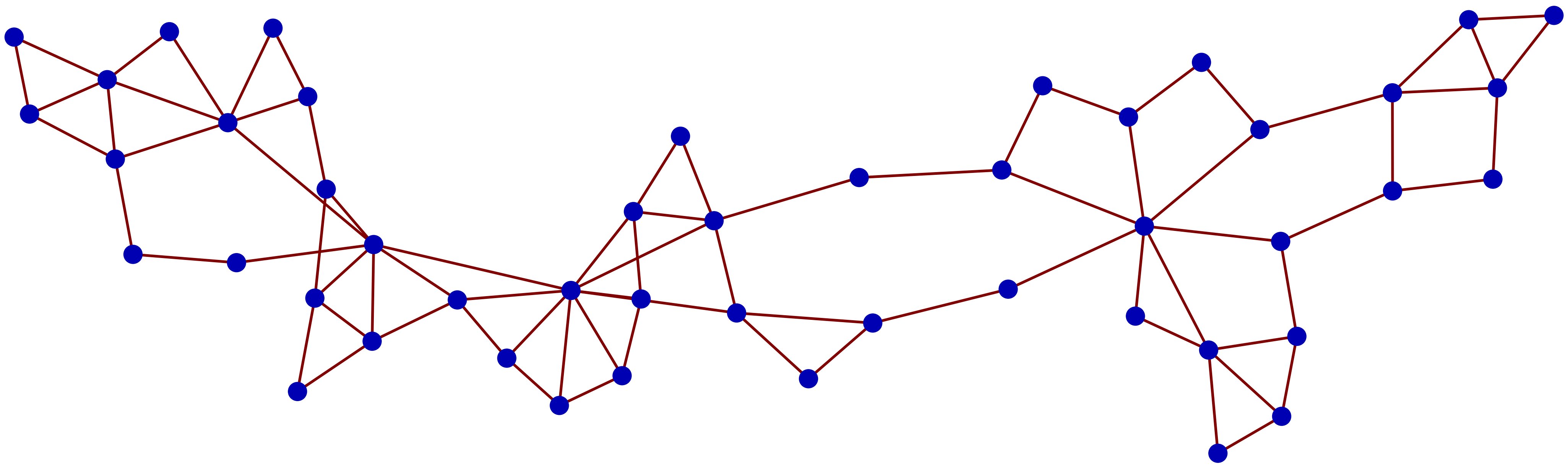

Figure: A uniform dissection of P_{45}.

Dissections seen as compact metric spaces

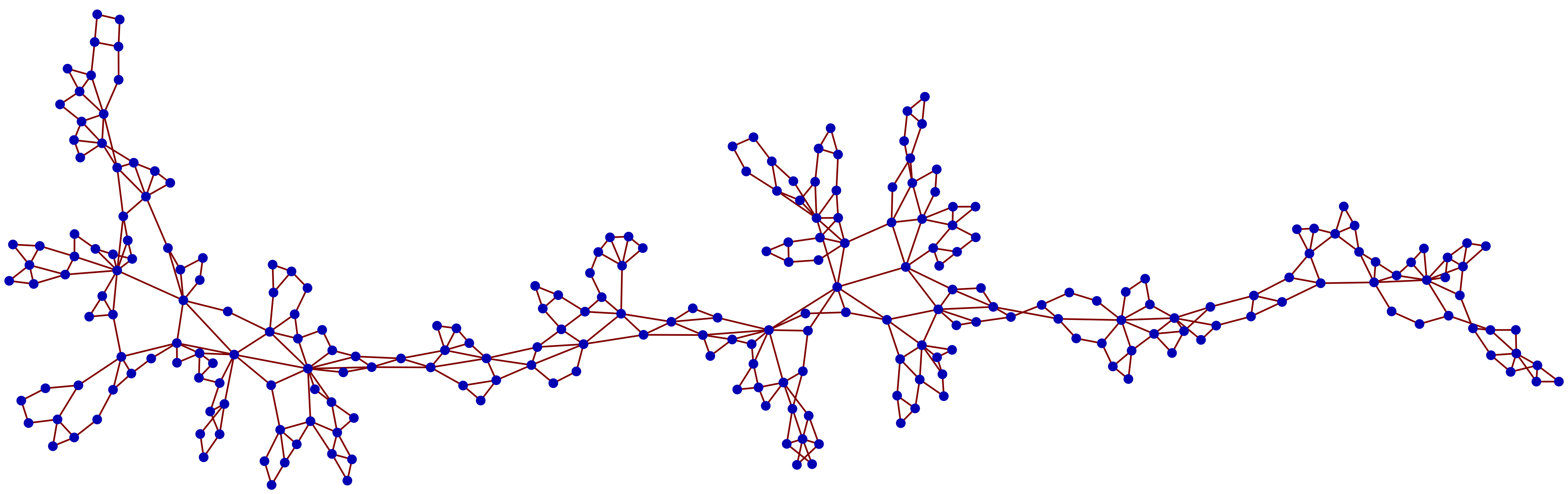

Figure: A uniform dissection of P_{260}.

Dissections seen as compact metric spaces

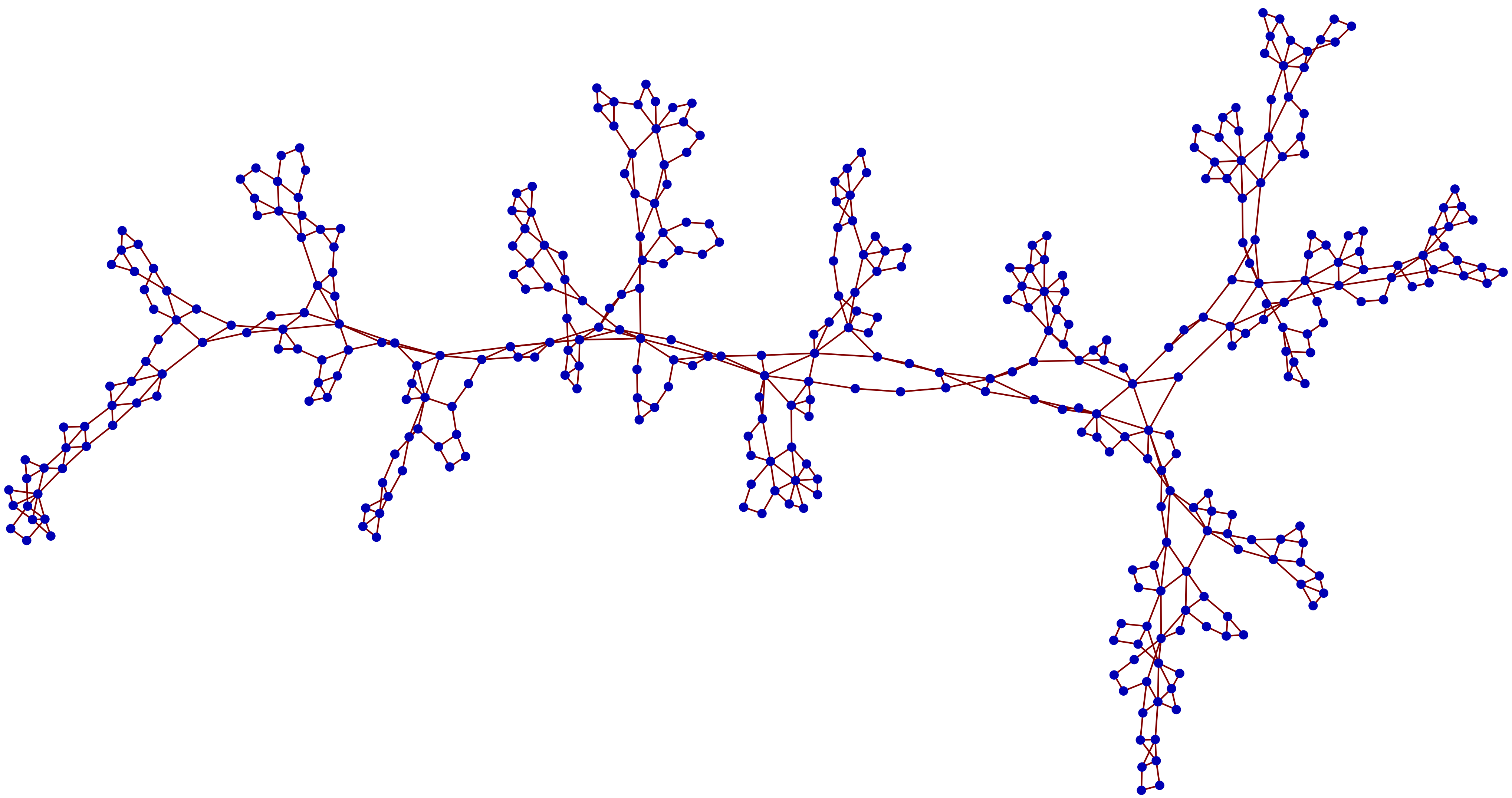

Figure: A uniform dissection of P_{387}.

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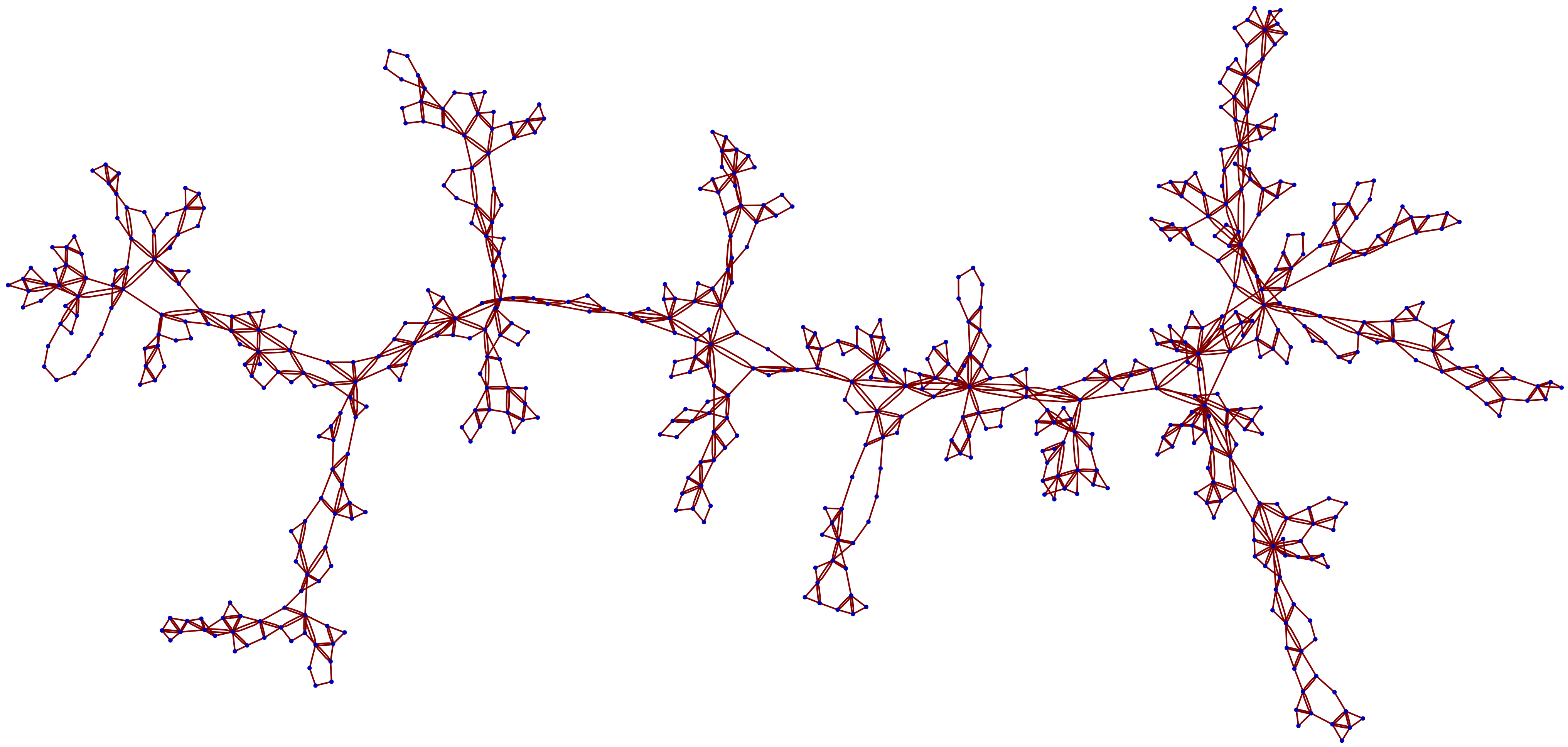

Figure: A uniform dissection of P_{637}.

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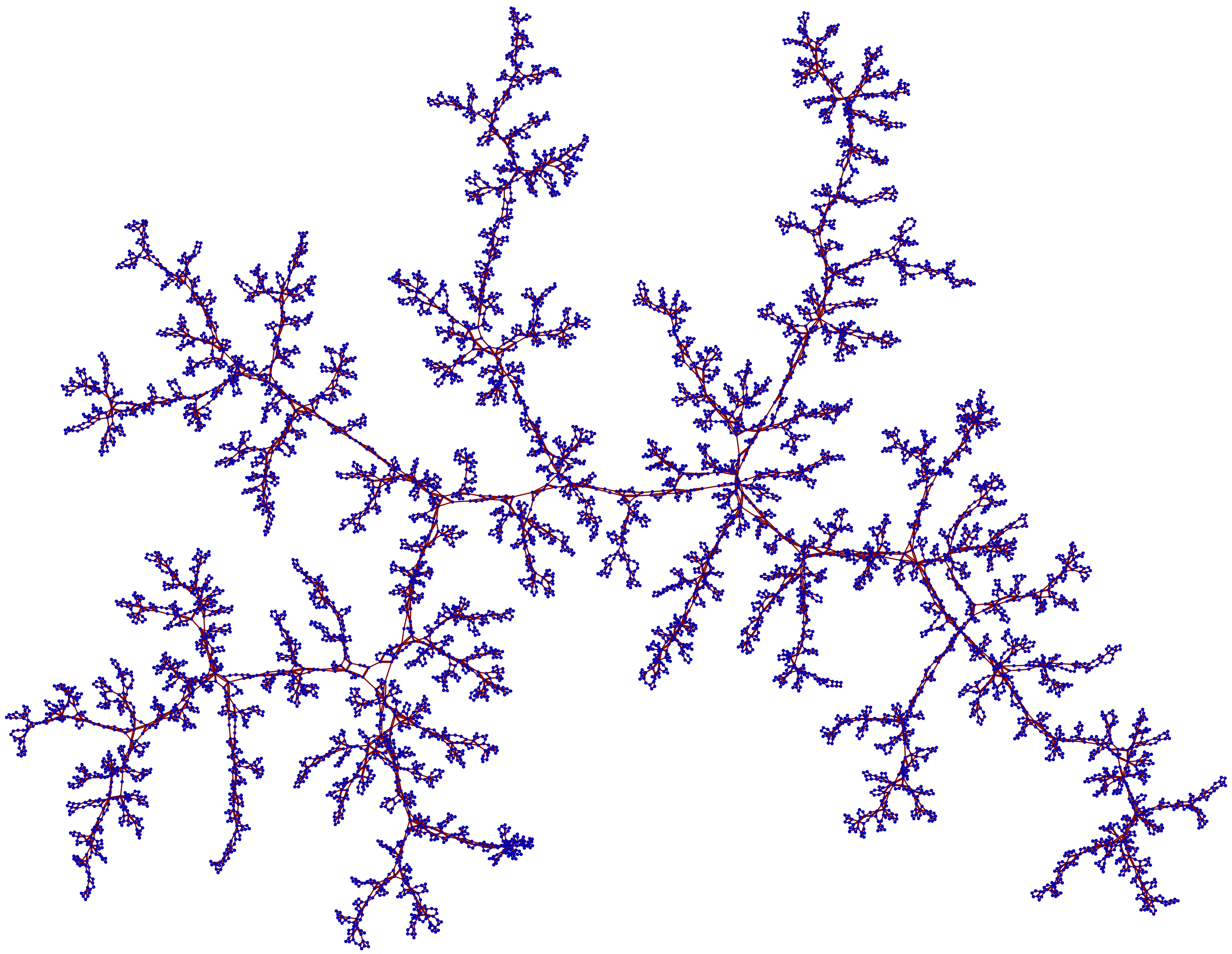

Figure: A uniform dissection of P_{8916}.

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I. Scaling limits of BGW trees (finite variance, 1991)

II. Scaling limits of BGW trees (infinite variance, 1998)

III. Plane non-crossing configurations (2012)

IV. Random maps (2004 – ?)

What does a “typical” random surface look like?

In dimension one

It is natural to view Brownian motion as a “typical” random path, describing the motion of a particle moving “uniformly at random”.

Brownian motion as a limit of discrete paths

Theorem (Donsker, 1951)

Let $(X_n)_{n \geqslant 1}$ *be a sequence of i.i.d. random variables such that* $\mathbb{E}[X_1] = 0$ *and* $\sigma^2 = \mathbb{E}\left[X_1{}^2\right] \in (0, \infty)$.

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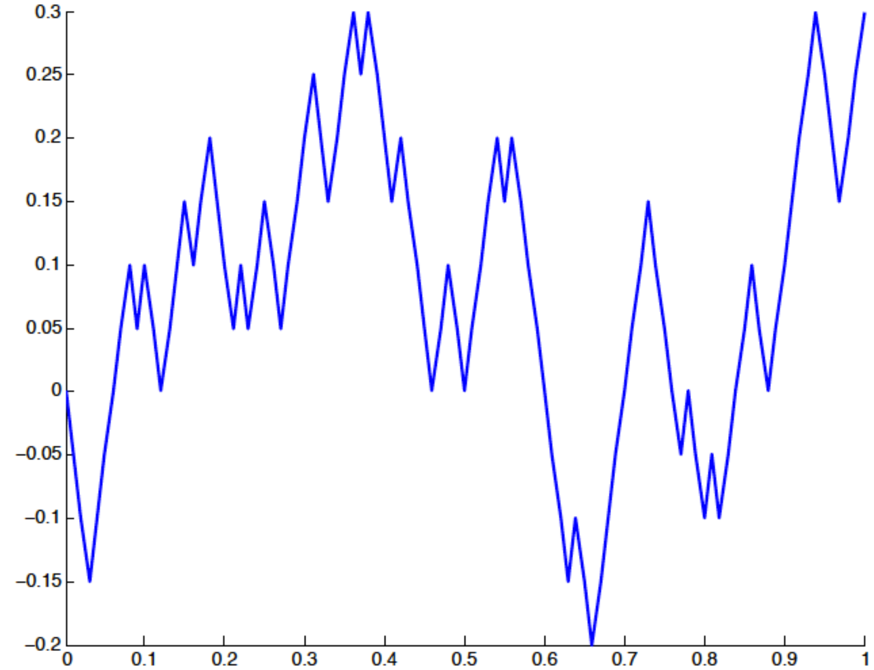

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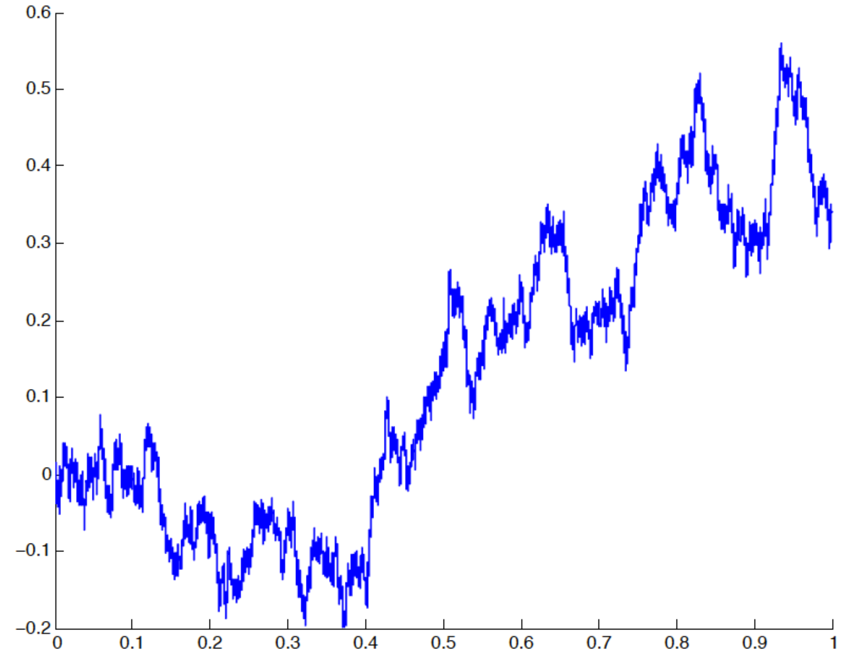

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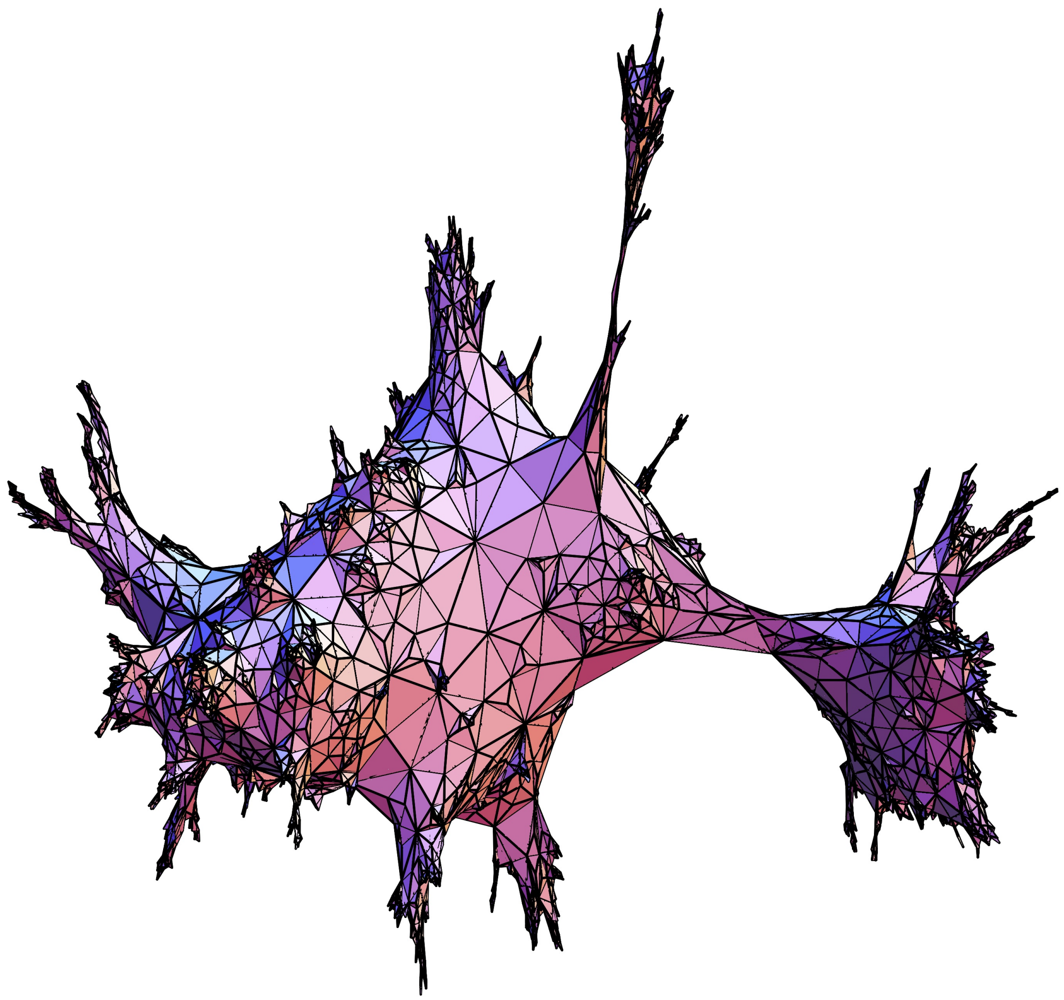

Figure: A large random triangulation (simulation by Nicolas Curien)

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Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles.

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(see Le Gall's proceeding at ICM '14 for more information and references)

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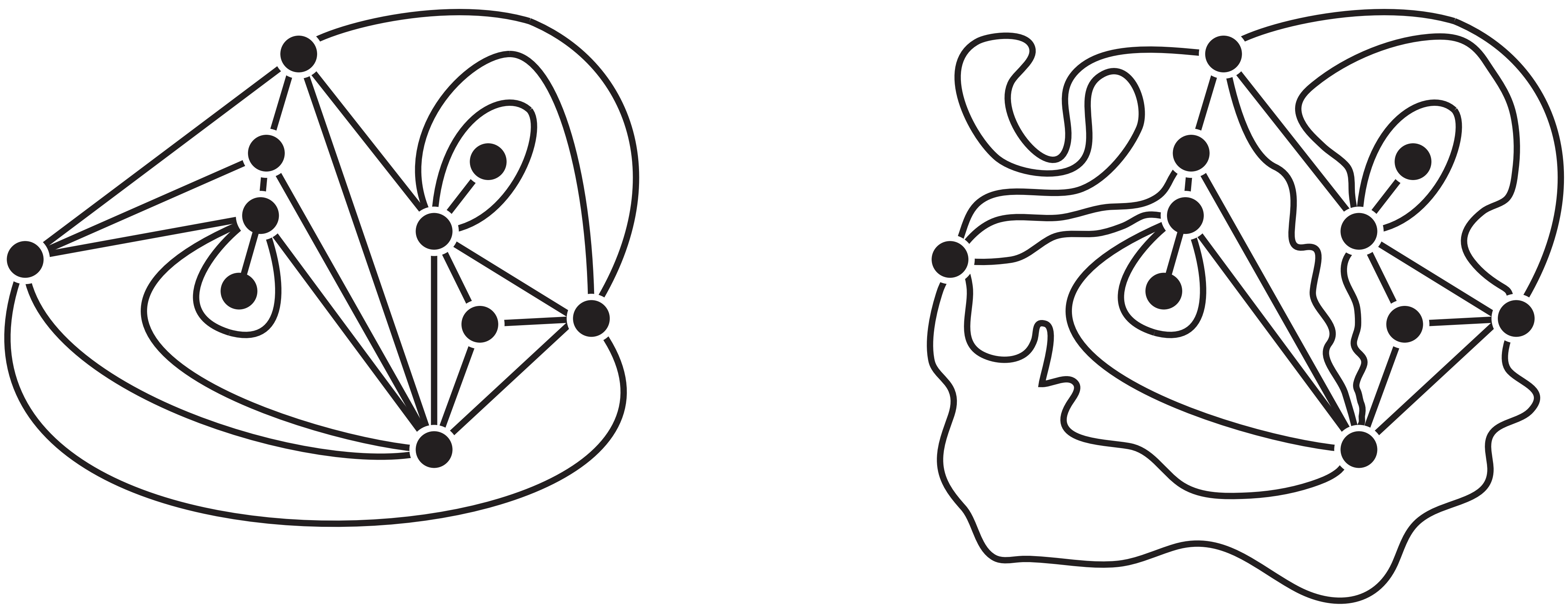

Figure: Two identical maps .

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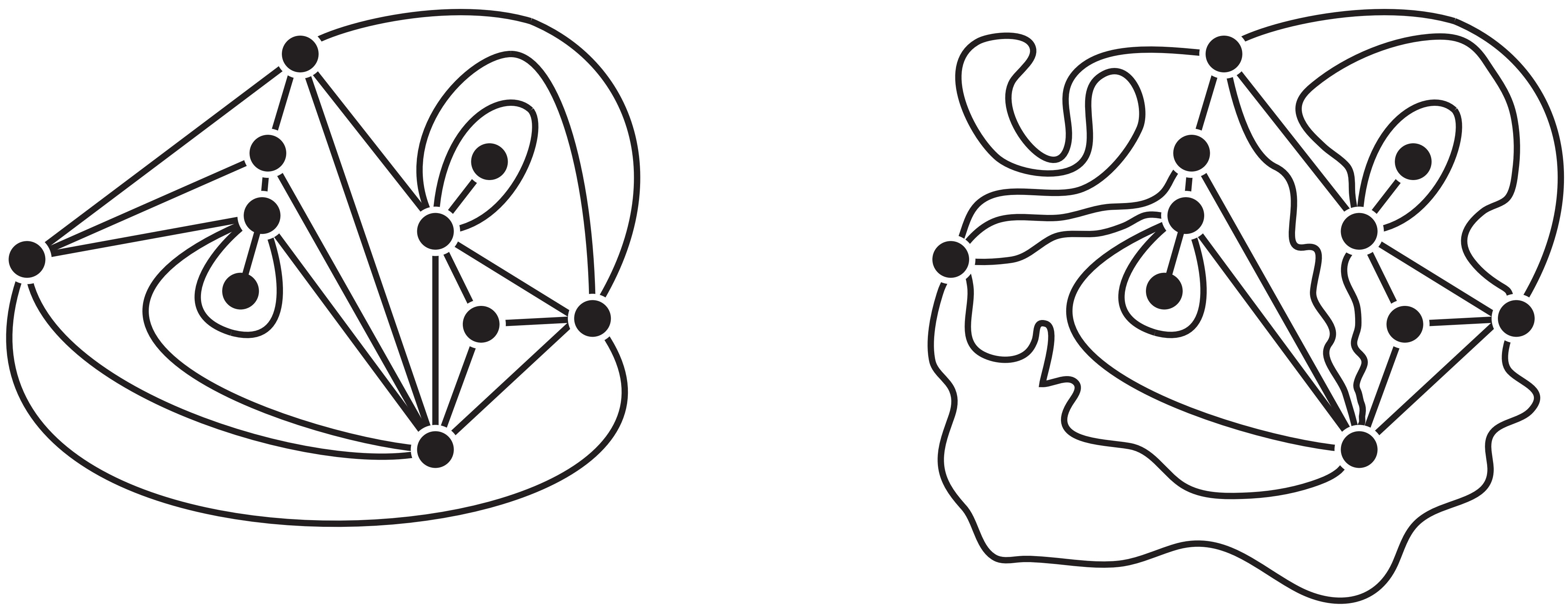

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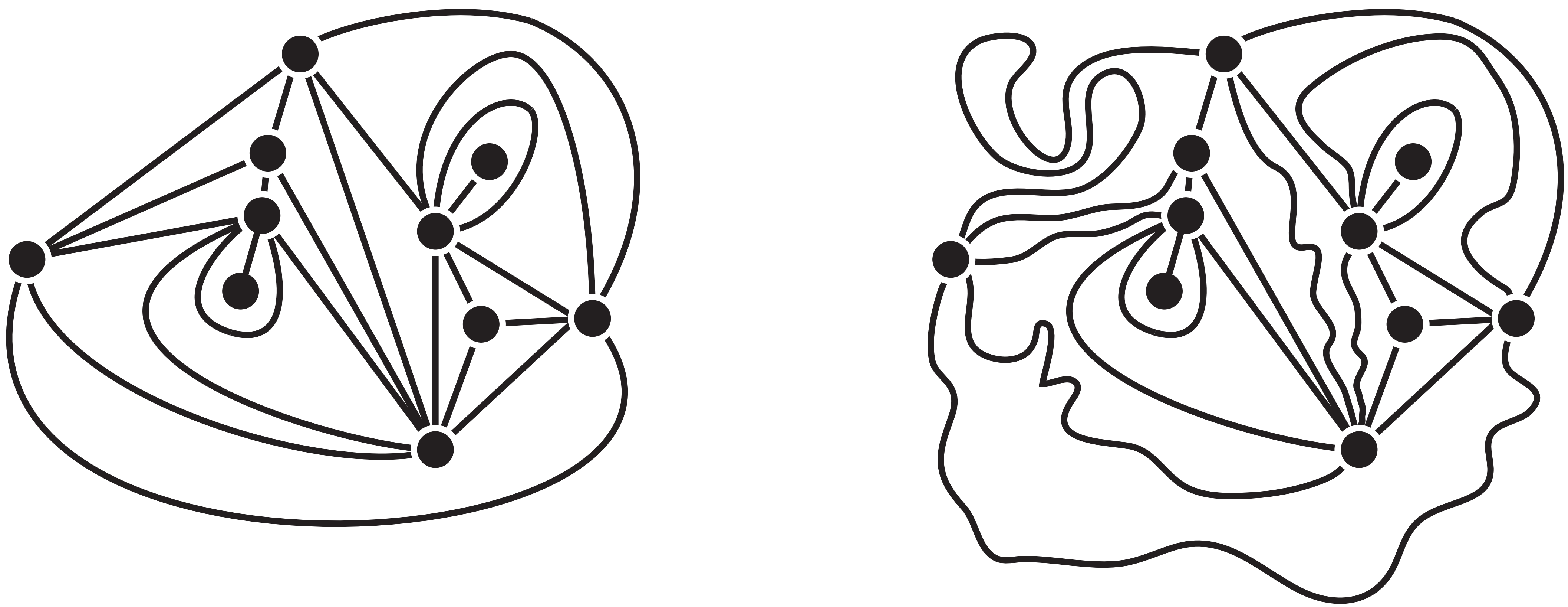

Figure: Two identical 3-angulations .

Why study maps?

- Combinatorics (Tutte starting in '60)
- Probability theory (model for a Brownian surface)
- Algebraix and geometric motivations Motivations (cf Lando–Zvonkine '04 *Graphs on surfaces and their applications*)
- Theoretical physics (connections with matrix integrals, 2D Liouville quantum gravity, KPZ formula.)

Scaling limits of large planar maps

Fix $p \geqslant 3$. Let M_n be a planar map, chosen uniformly at random among all p-angulations with n faces.

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There exists a constant $c_p > 0$ *and a random compact metric space* (m_∞, D^*), *called the Brownian map, such that the convergence*

$$\left(V(M_n), c_p n^{-1/4} d_{gr}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (m_\infty, D^*)$$

holds in distribution in the space of isometry classes of compact metric spaces equiped with the Gromov–Hausdorff distance.

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⇝ 3/2-stable spectrally positive Lévy processes and 3/2-stable trees play a crucial role in the study of these maps, see the talk of Nicolas Curien next week.