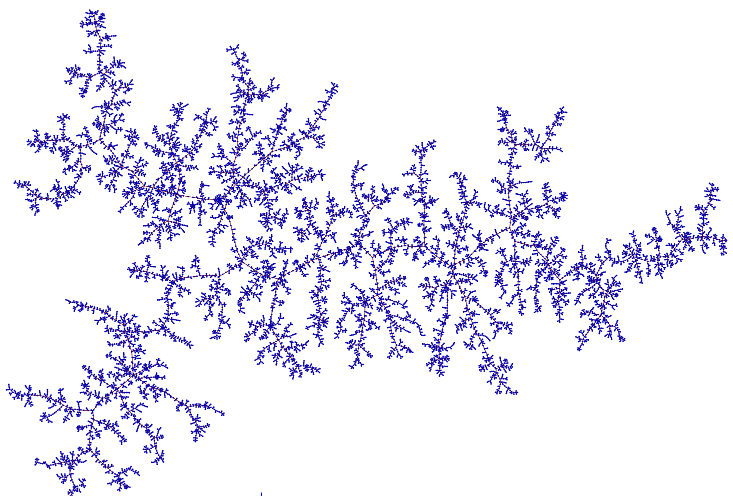
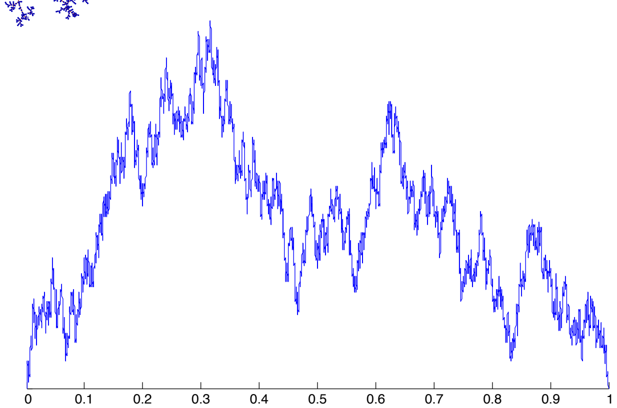
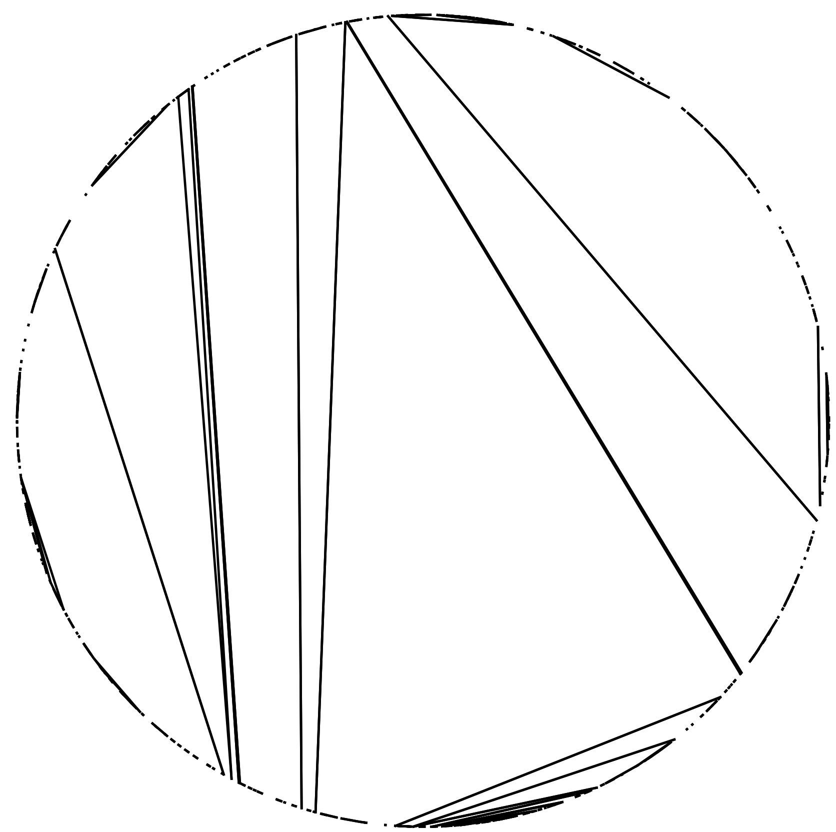
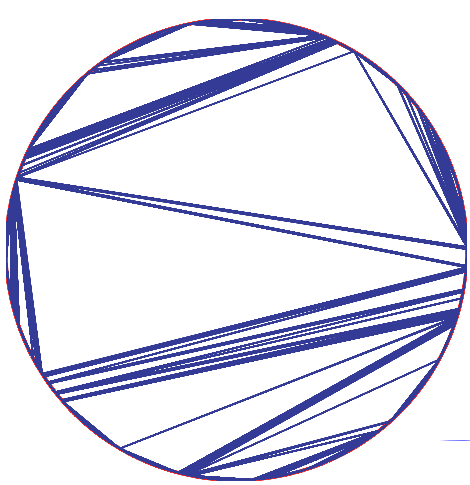
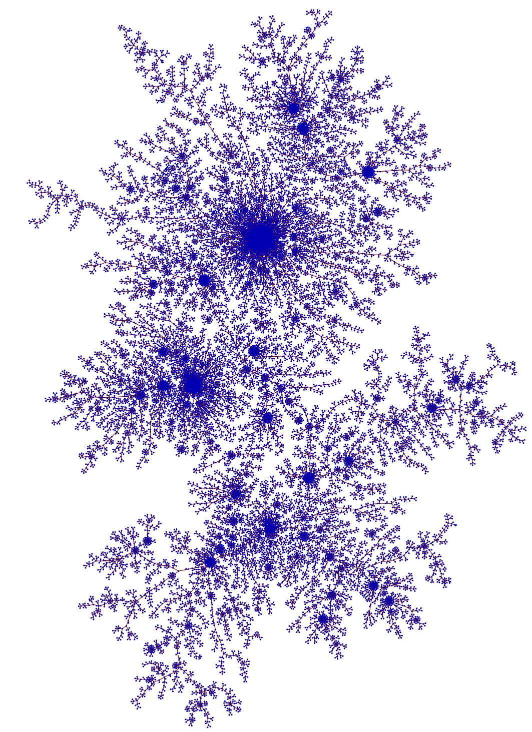
Asymptotic behavior of large random discrete structures

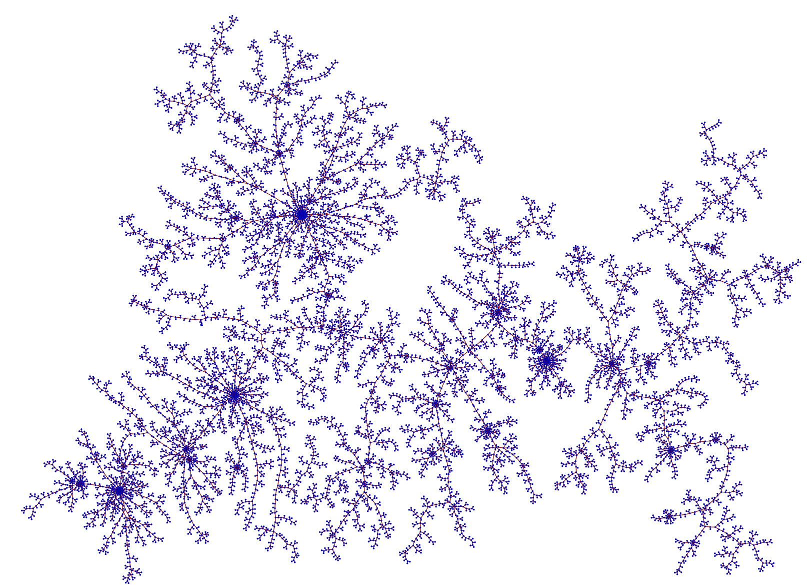












Igor Kortchemski
(with Valentin Féray)
CNRS & École polytechnique

Journées de combinatoire de Bordeaux – Février 2019

Questions: minimal factorizations

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$$\mathfrak{M}_n = \{(\tau_1, \dots, \tau_{n-1}) \text{ transpositions } : \tau_1 \tau_2 \cdots \tau_{n-1} = (1, 2, \dots, n)\}$$

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$\leadsto$ Question: for n large, what does a typical minimal factorization look like?

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To answer this question, a possibility is to find a continuous object X such that $X_n \to X$ as $n \to \infty$.

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- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging to X, then X_n and Y_n "roughly" have the same properties for n large.

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- *In what space do the objects live?* Here, a metric space (Z, d) (complete separable).
- *What is the sense of this convergence when these objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \quad \underset{n \to \infty}{\longrightarrow} \quad \mathbb{E}\left[F(X)\right]$$

for every continuous bounded function $F : Z \to \mathbb{R}$.

Outline

I. Trees

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$\rightsquigarrow$ Probability: trees are elementary pieces of various models of random graphs, having rich probabilistic properties.

Plane trees

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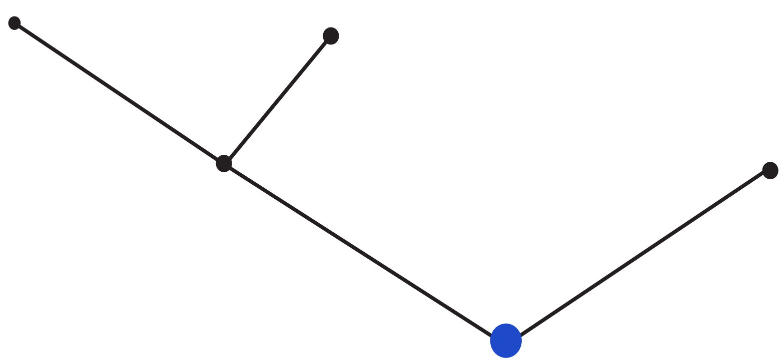

Figure: Two different plane trees

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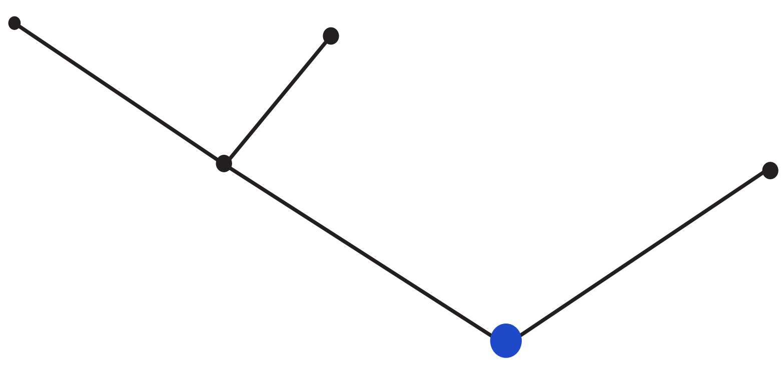
Let $\mathfrak{X}_n$ be the set of all plane trees with n vertices.



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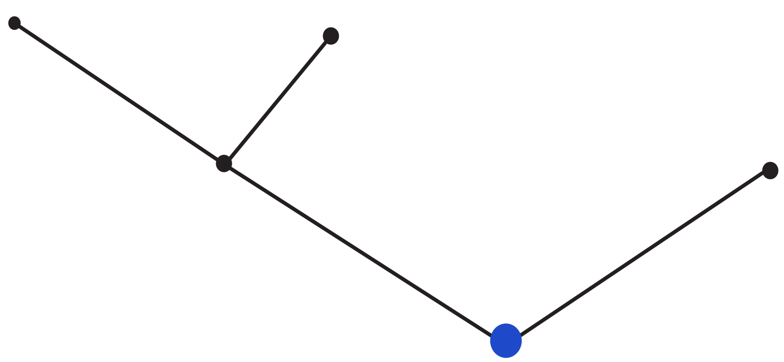

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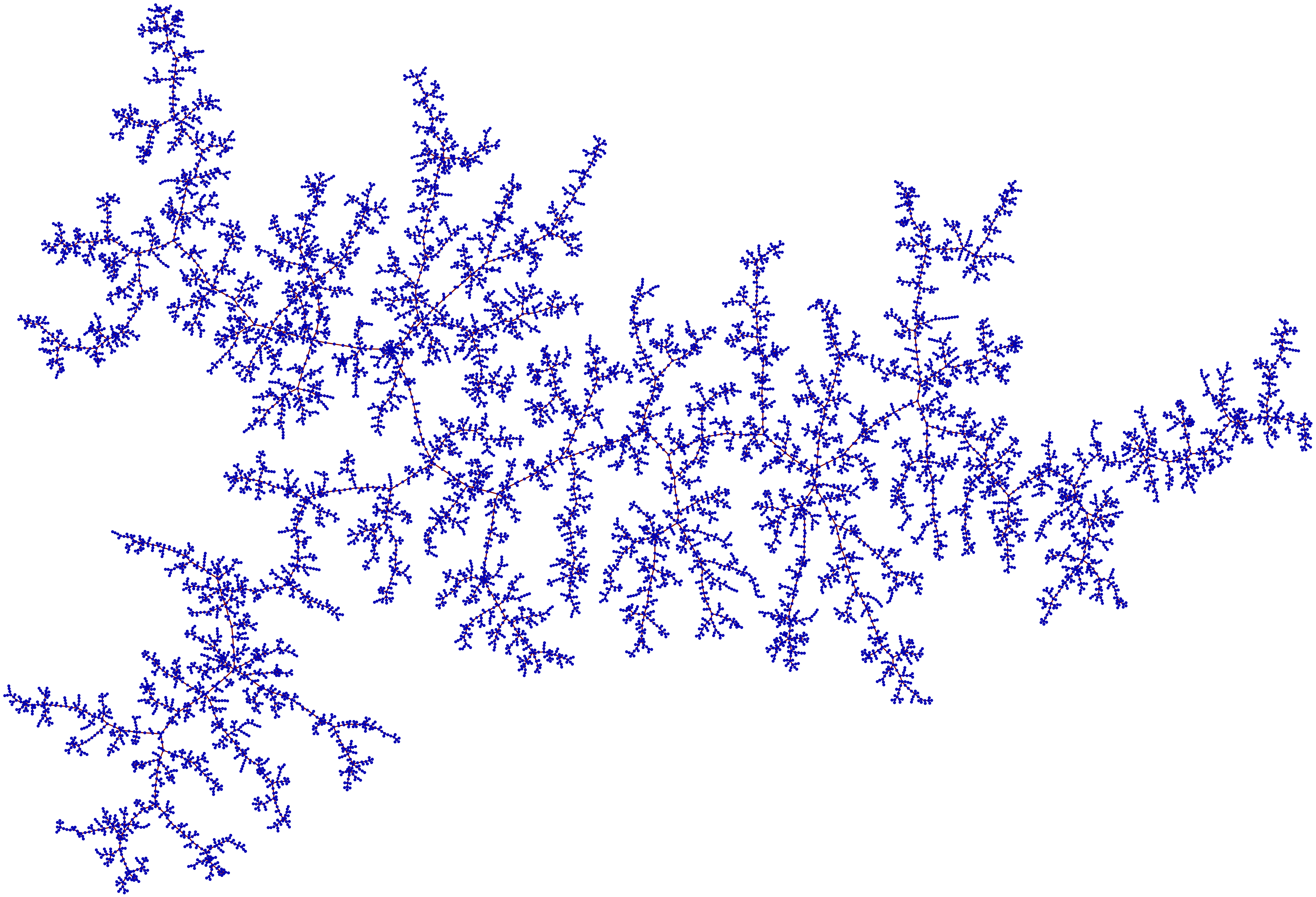
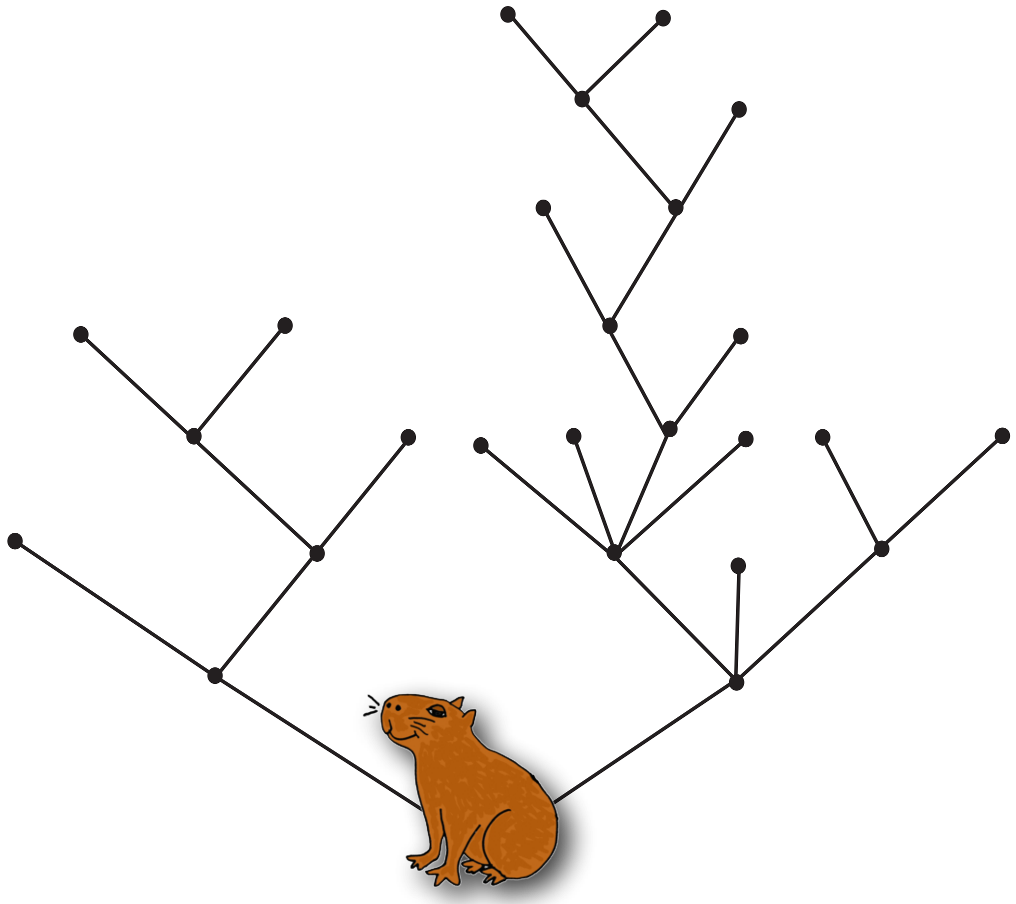
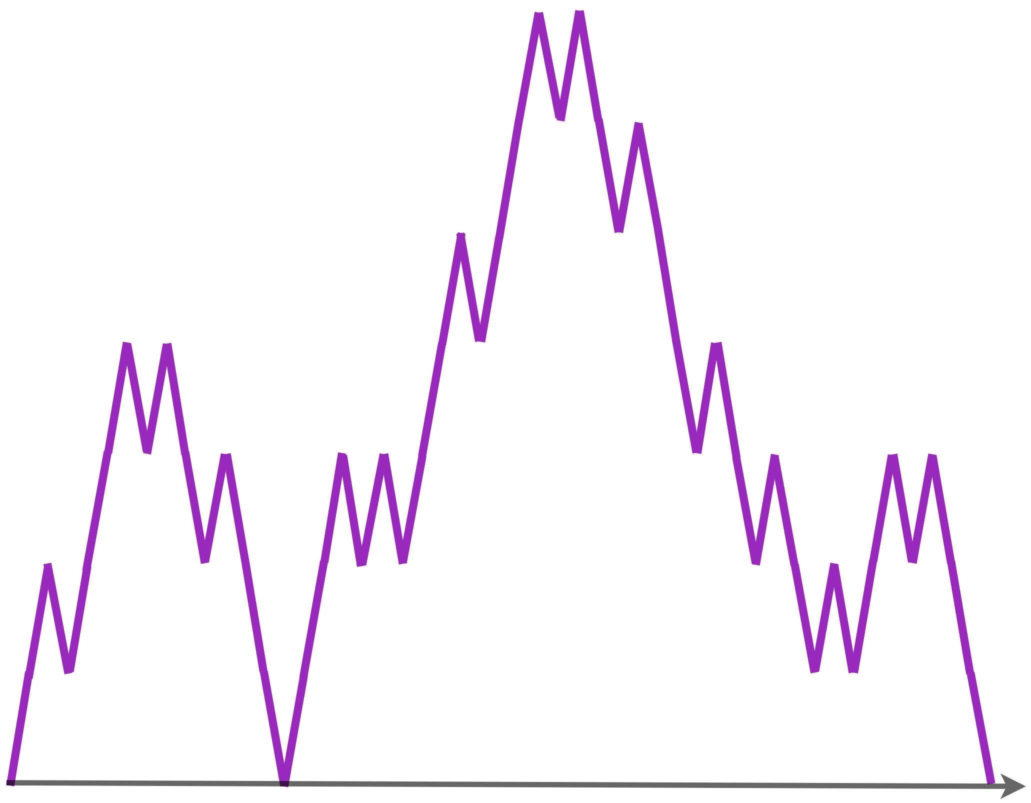


Coding a tree by its contour function

Code a tree τ by its contour function $C(\tau)$:





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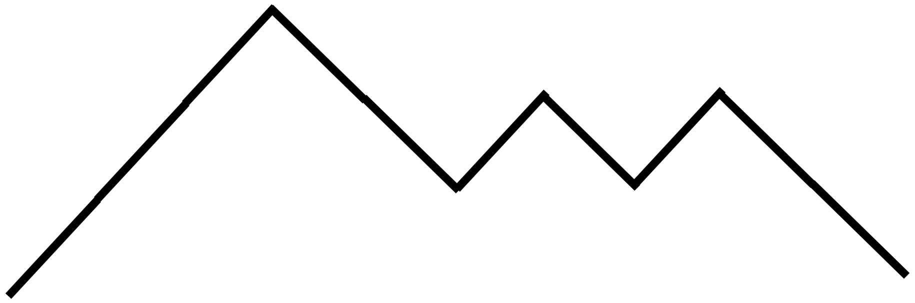
Knowing the contour function, it is easy to reconstruct the tree:



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Theorem (Aldous '93)

We have:

$$\left(\frac{1}{\sqrt{2n}} C_{2nt}(t_n)\right)_{0 \leqslant t \leqslant 1} \xrightarrow[n \to \infty]{(d)} (\mathbb{e}(t))_{0 \leqslant t \leqslant 1},$$

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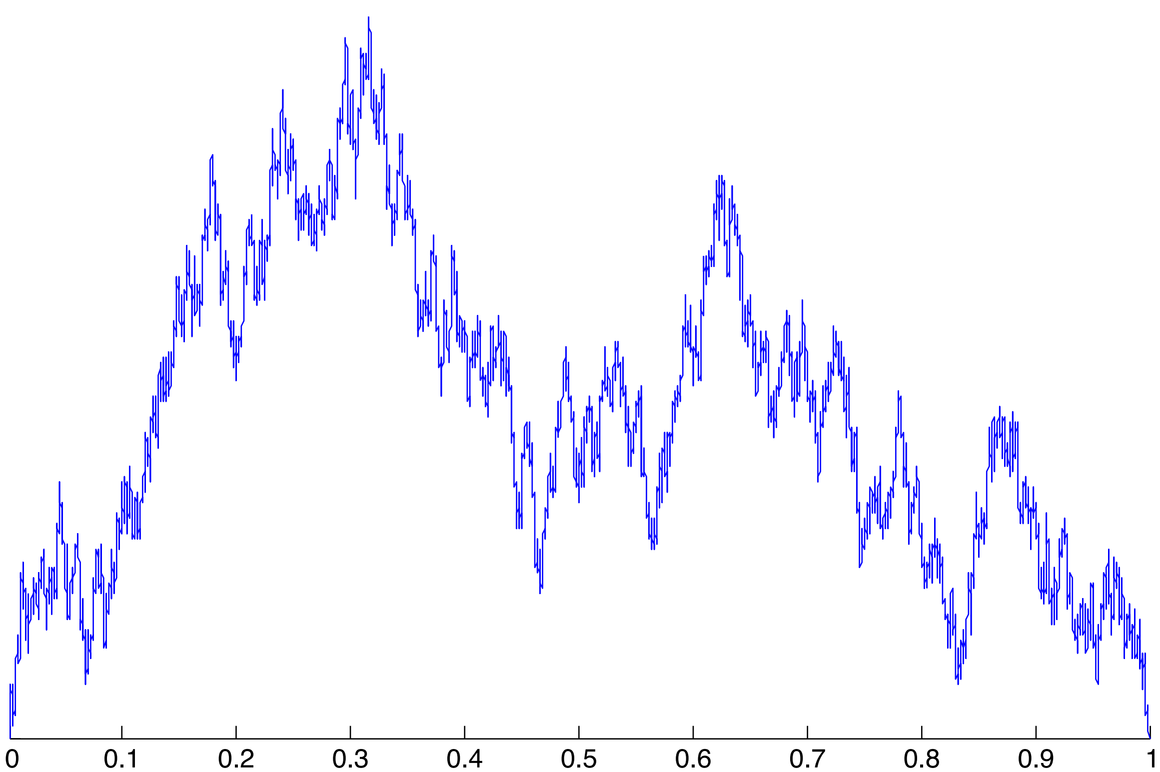

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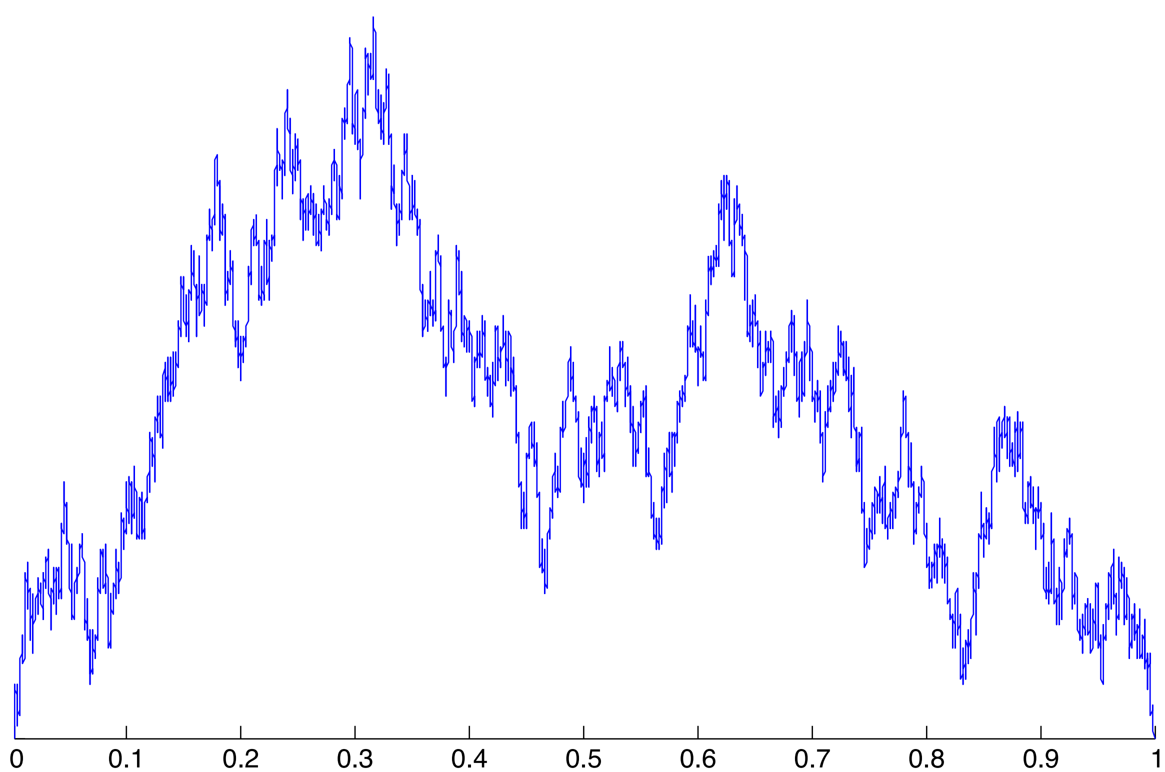

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⟿ Idea: t_n is a (conditioned) random walk, use (a conditioned) Donsker's invariance principle.

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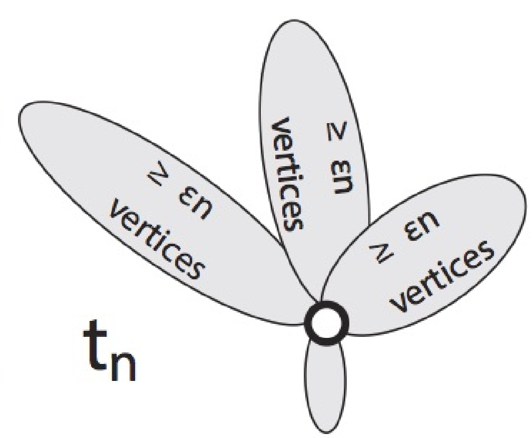
$\rightsquigarrow$ Consequence 2: for every $\varepsilon > 0$,

$$\mathbb{P}\left(\text{there exists a vertex of } t_n \text{ with 3 grafted subtrees of sizes } \geqslant \varepsilon n\right) \to 0.$$

$\mathbb{P}\Big($ 

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- different families of tree-like structures: stack triangulations (Albenque & Marckert), graphs from subcritical classes (Panagiotou, Stufler & Weller), dissections (Curien, Haas & K), various maps (Janson & Stefánsson, Bettinelli, Caraceni, K & Richier).

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II. Triangulations & dissections

III. Minimal factorizations

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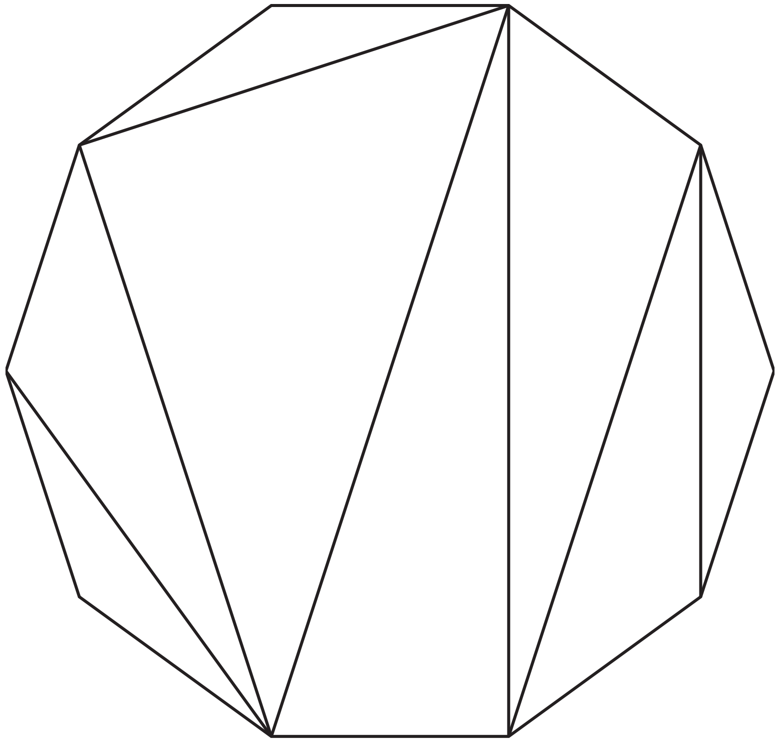

Figure: A triangulation of $\mathcal{X}_{10}$.

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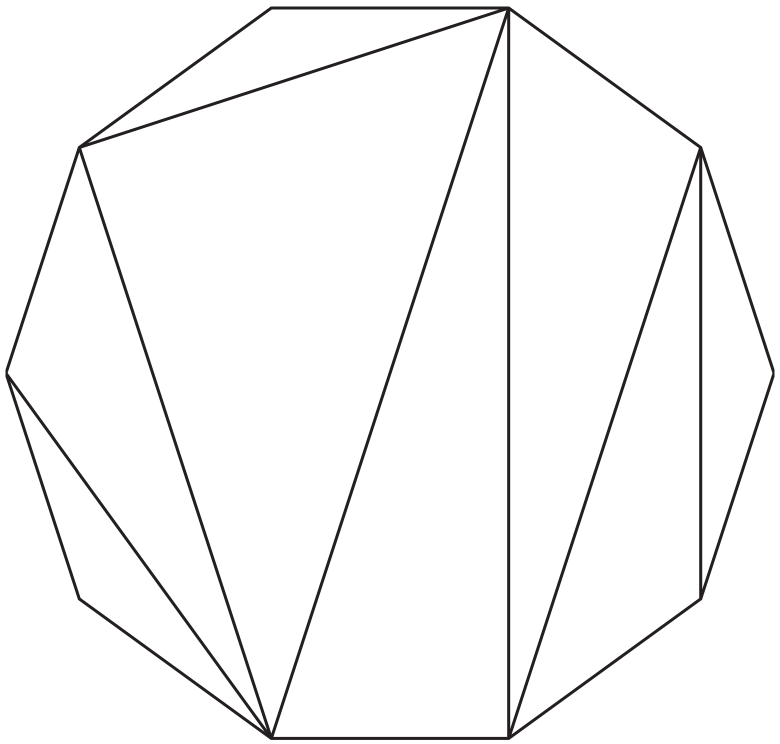

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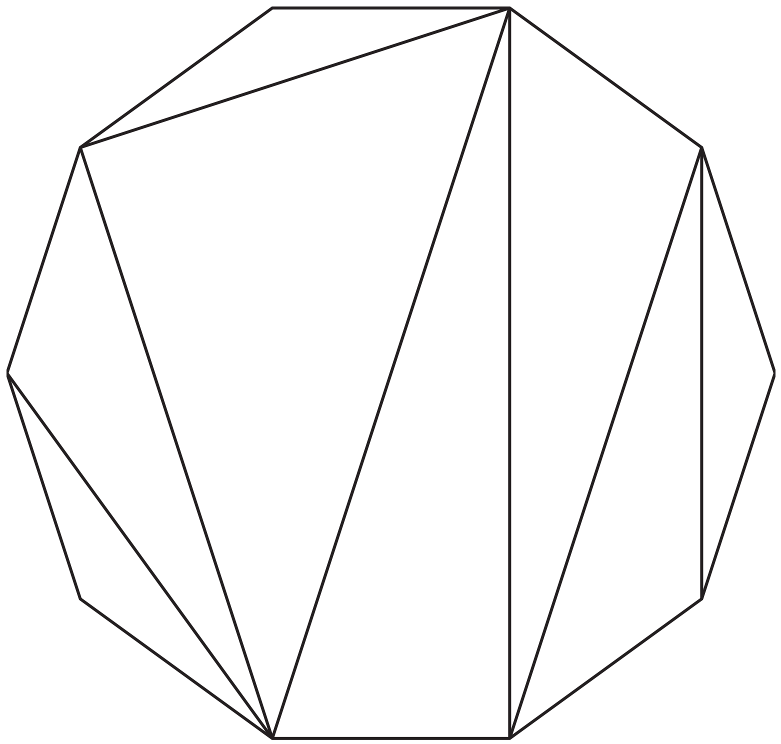

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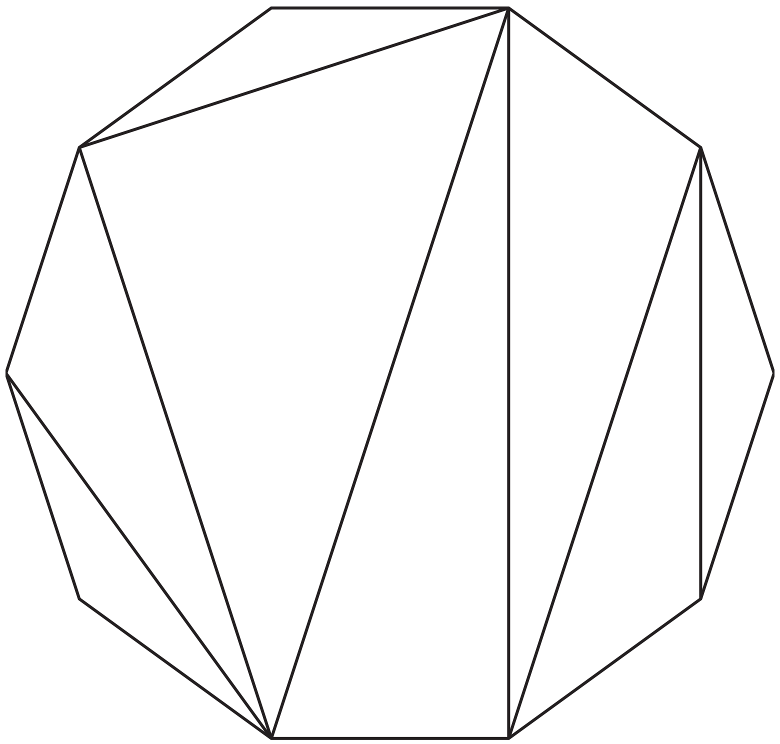

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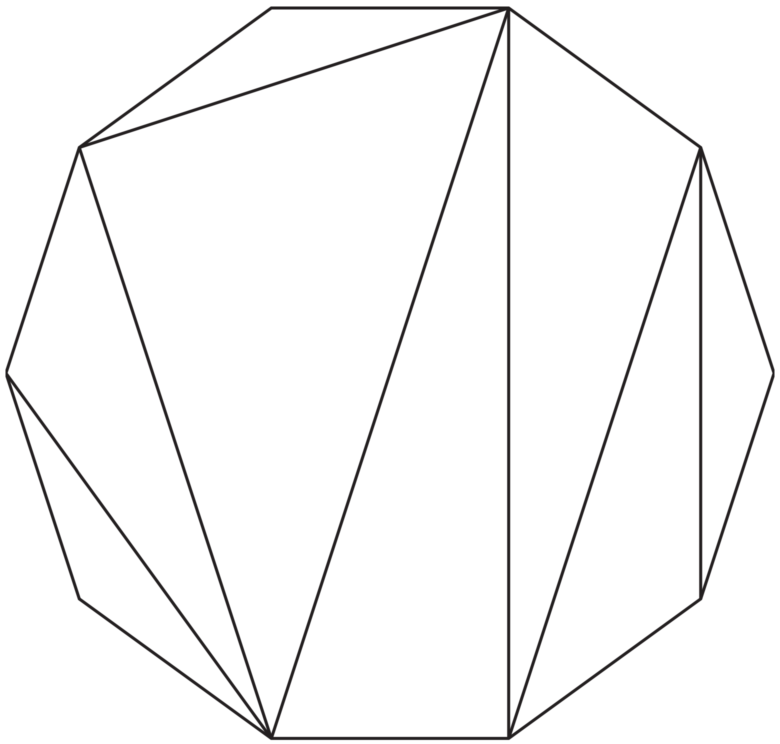

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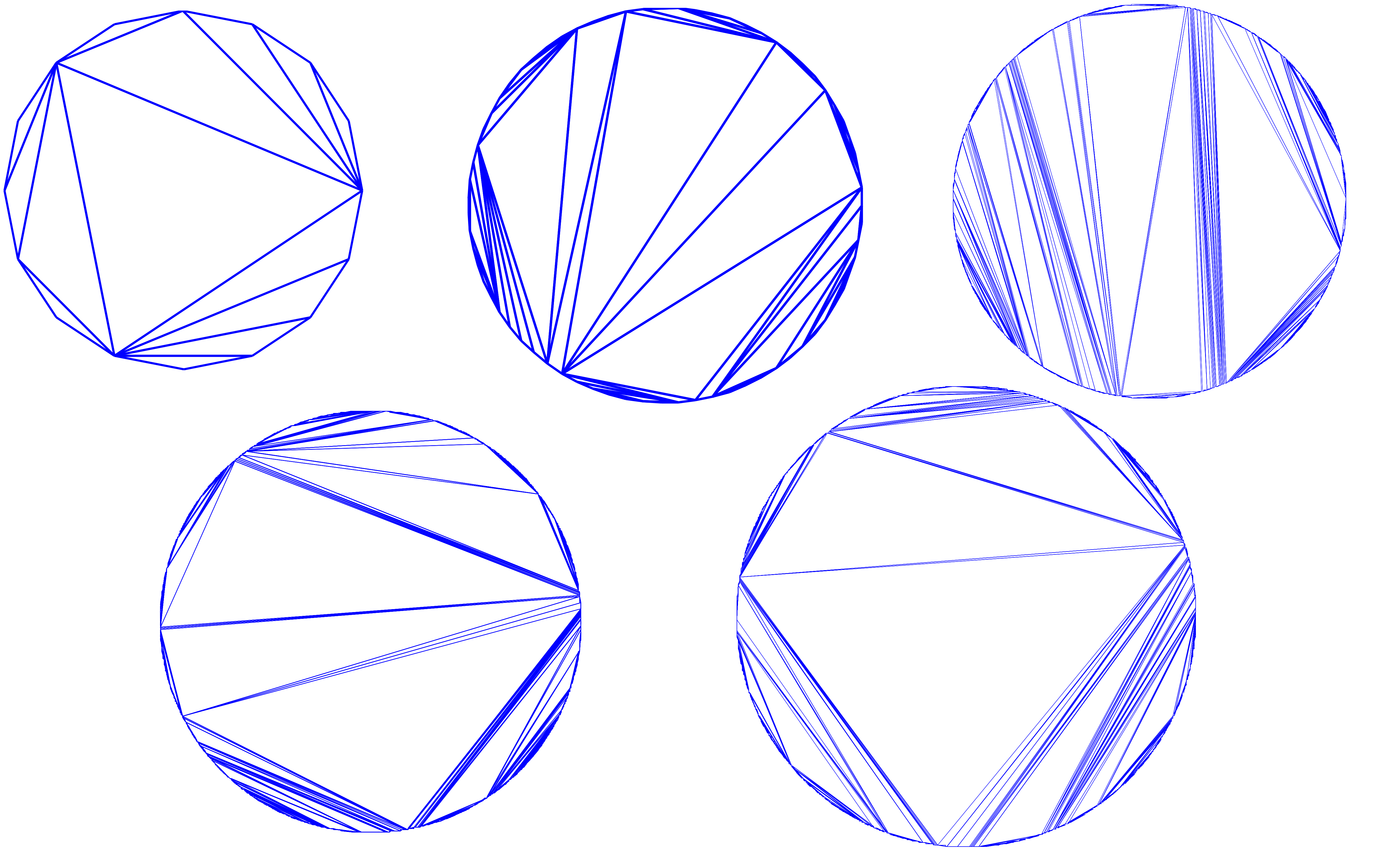
$\leadsto$ Question: What does a large typical triangulation look like?

Typical triangulations



What space for triangulations?

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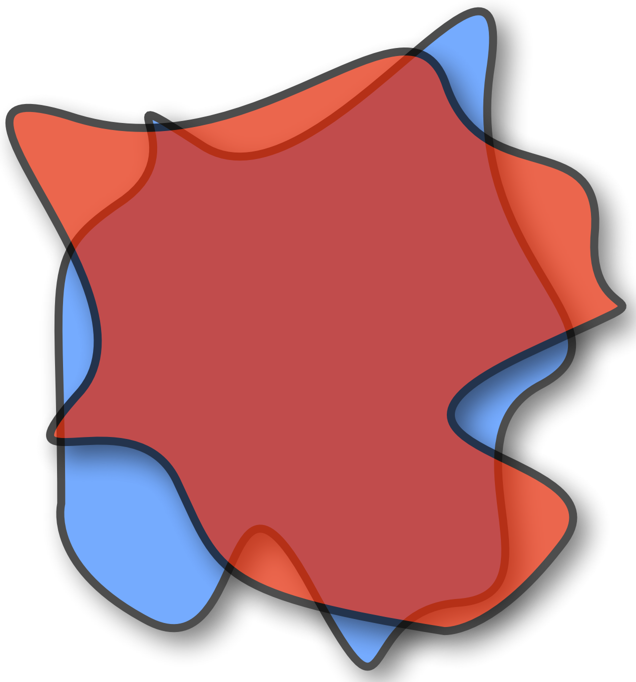

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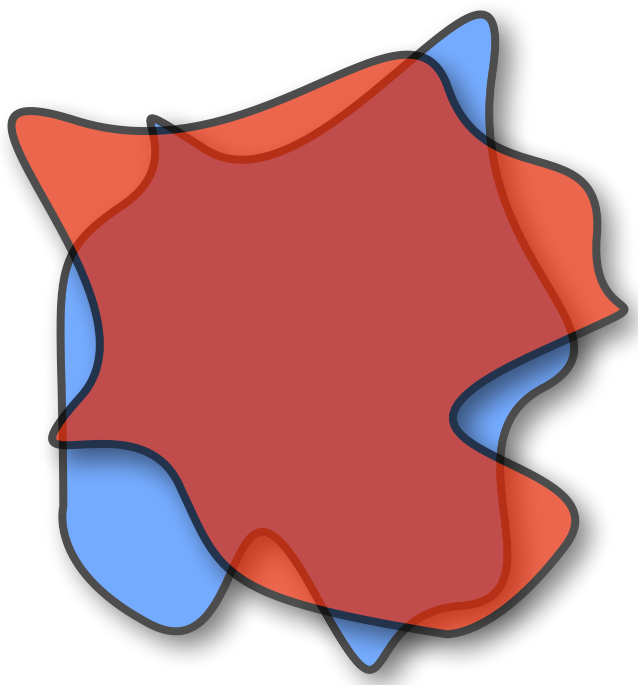

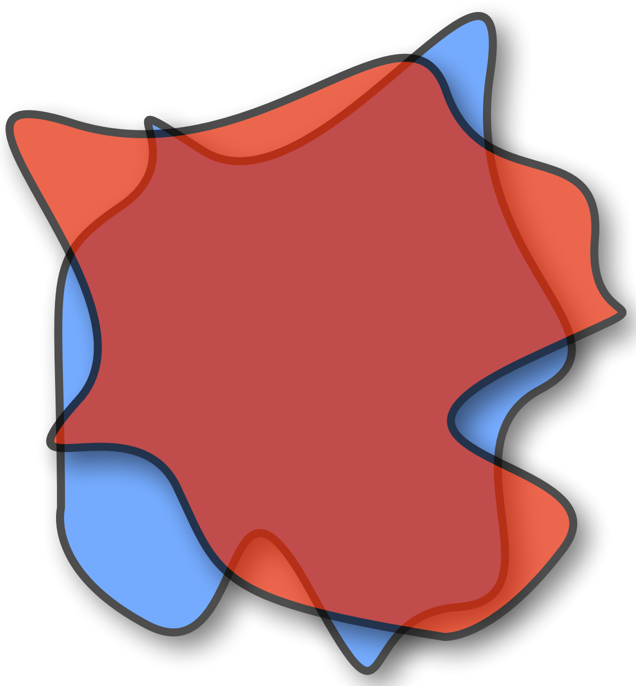

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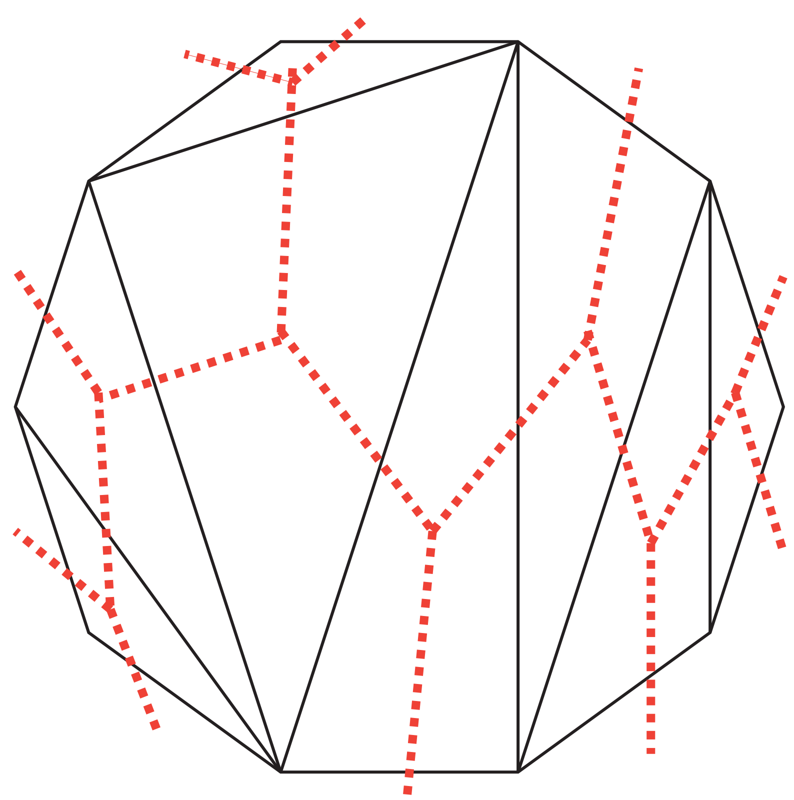

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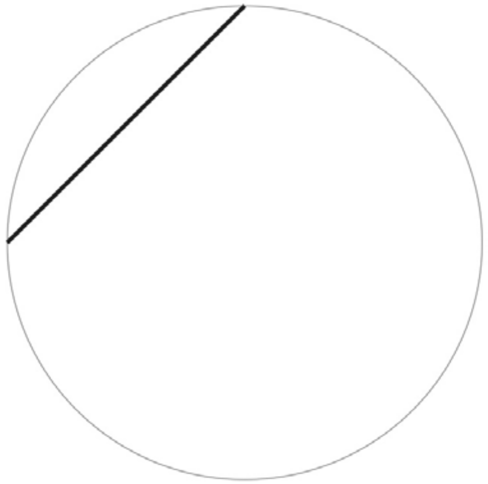

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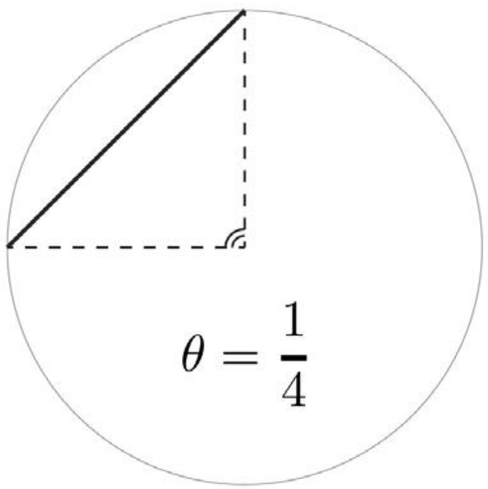

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(Aldous, Devroye–Flajolet–Hurtado–Noy–Steiger)

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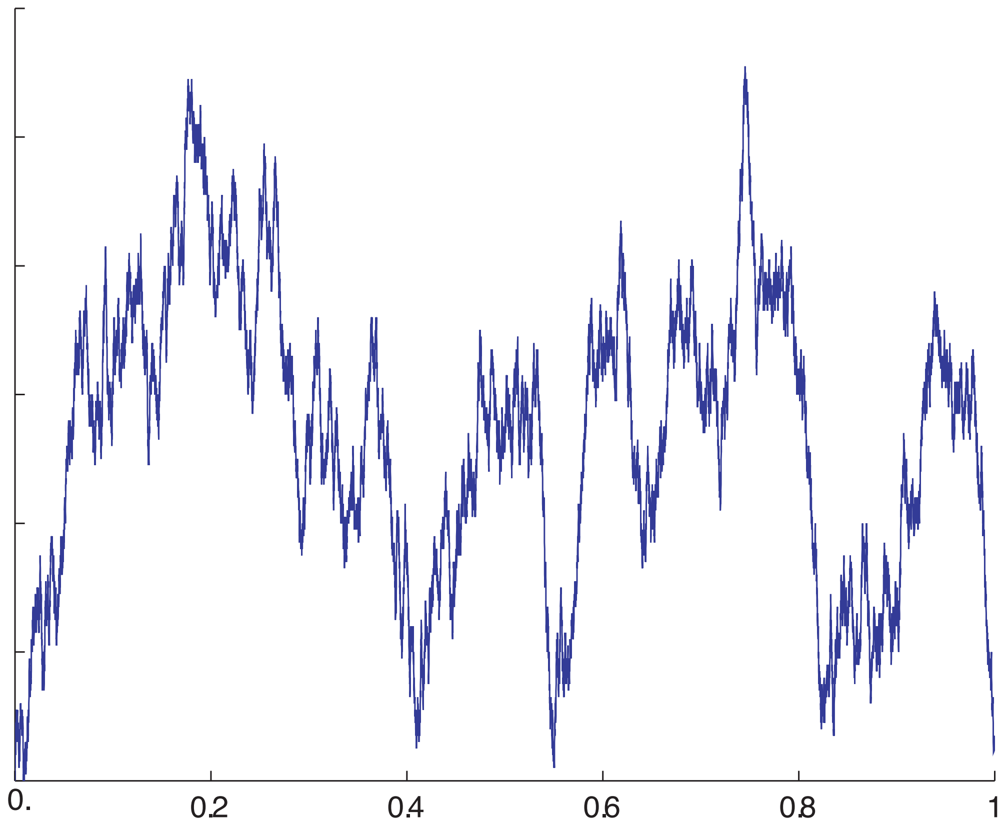

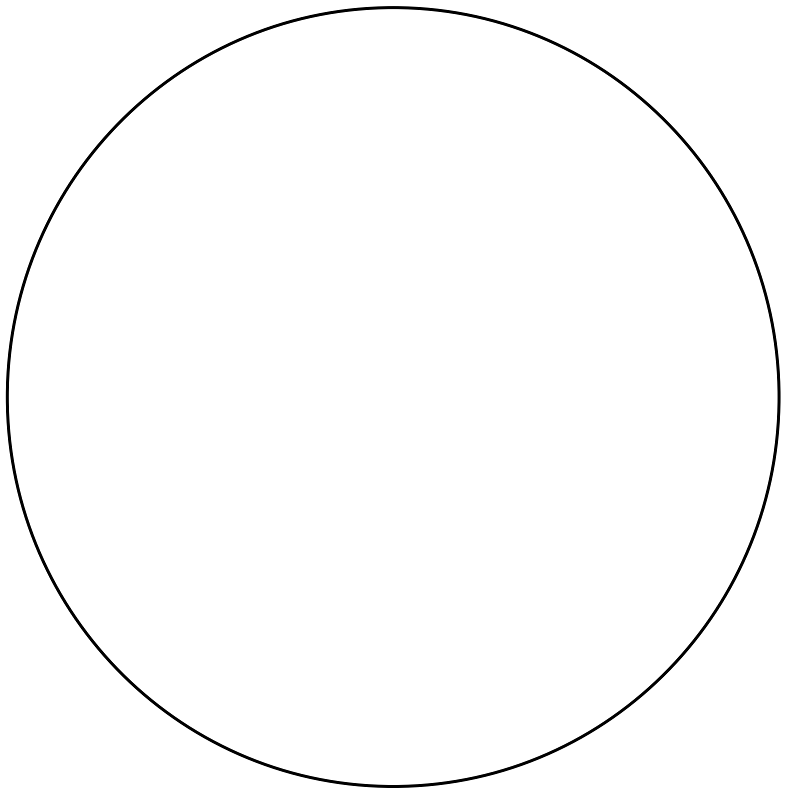

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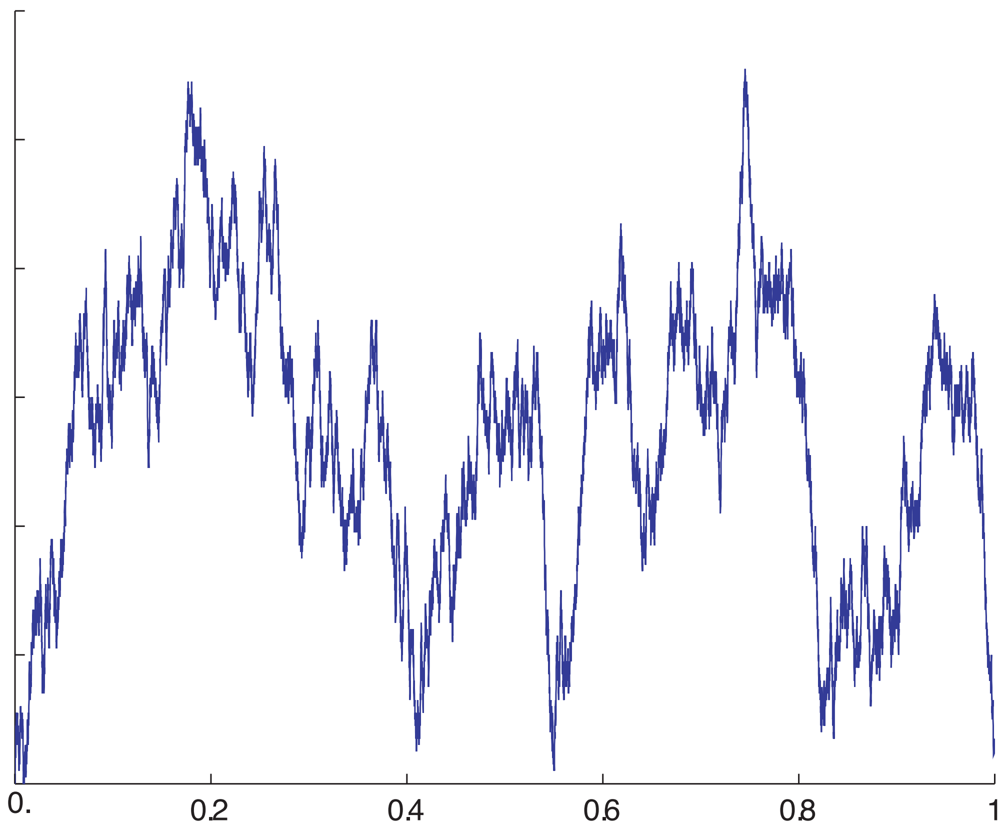

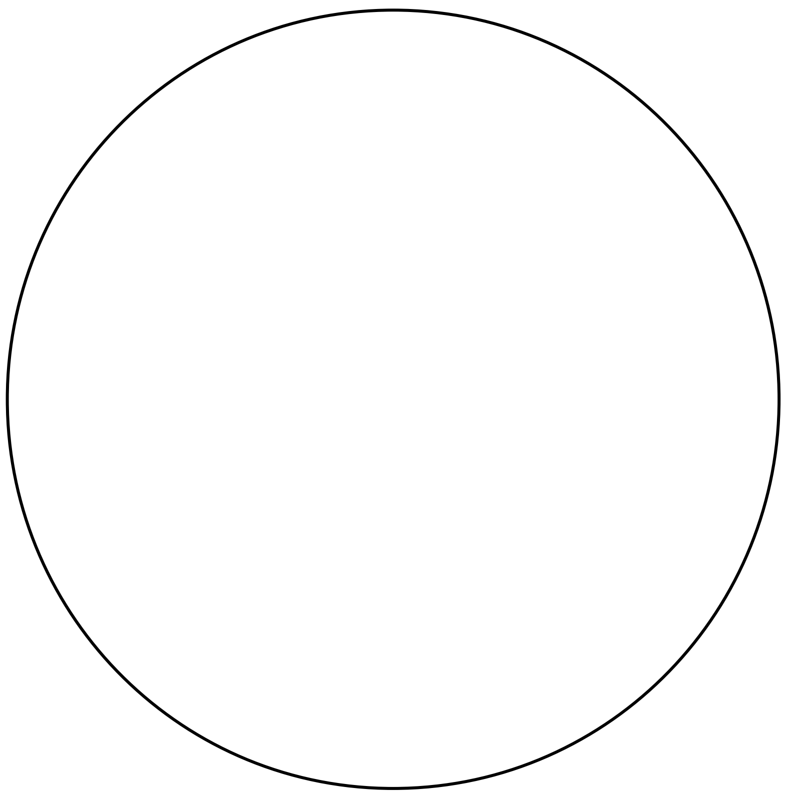

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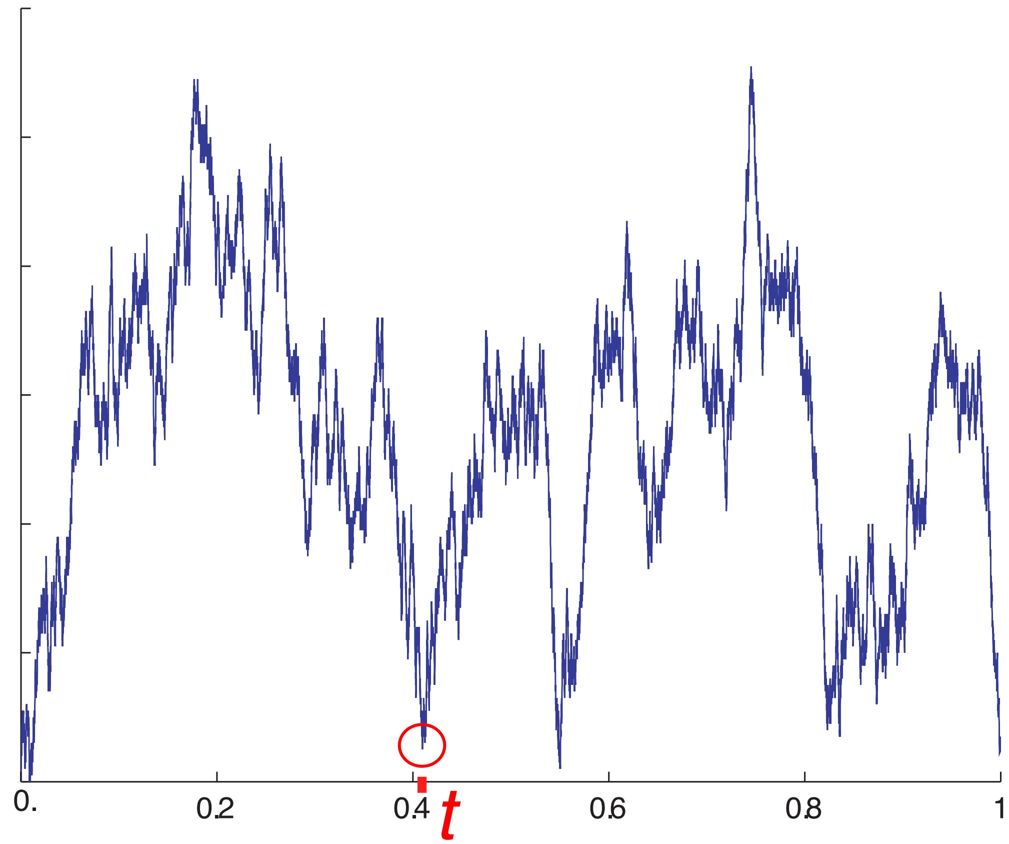

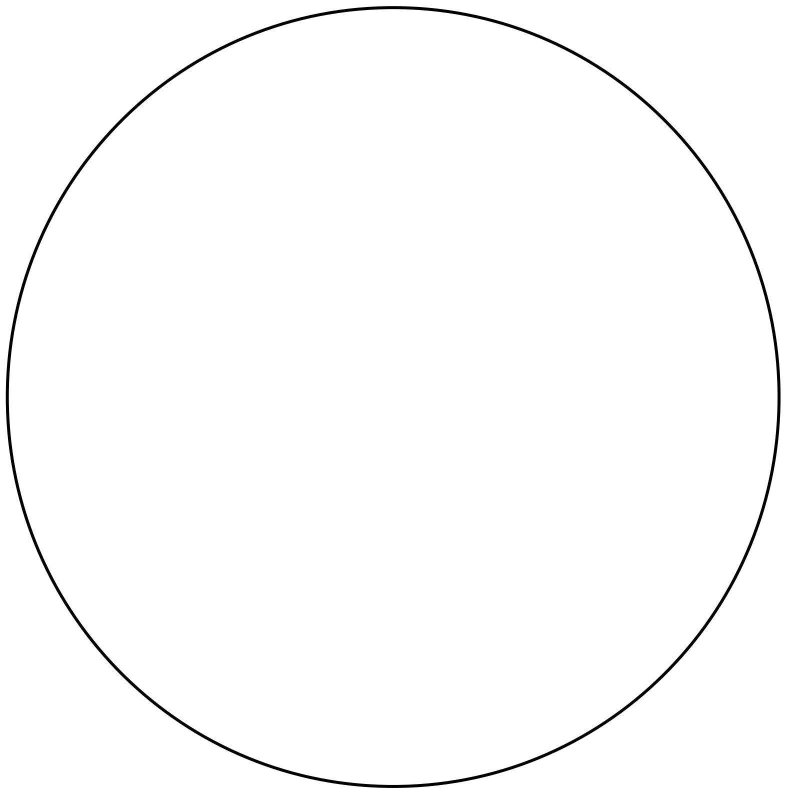

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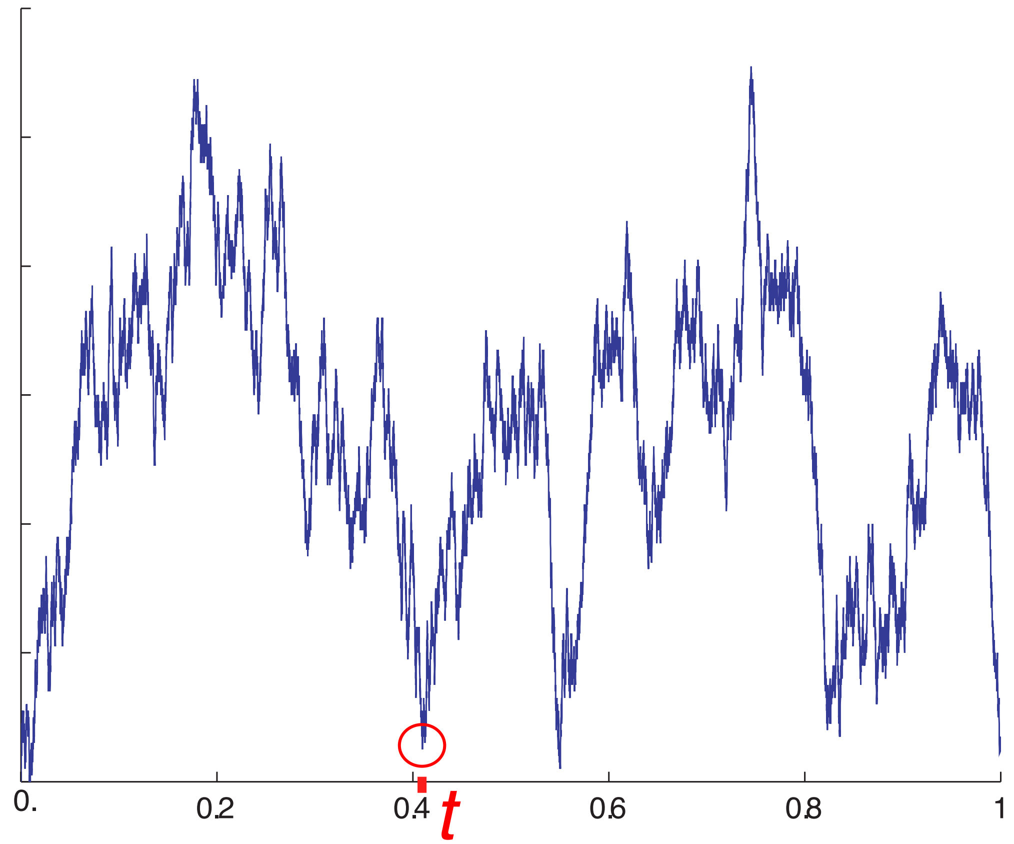

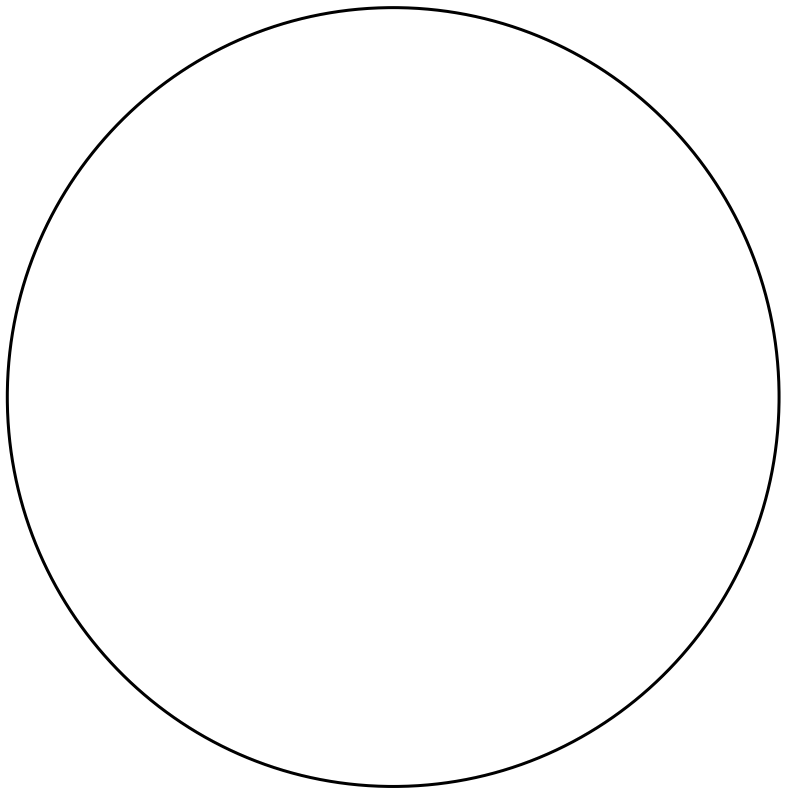

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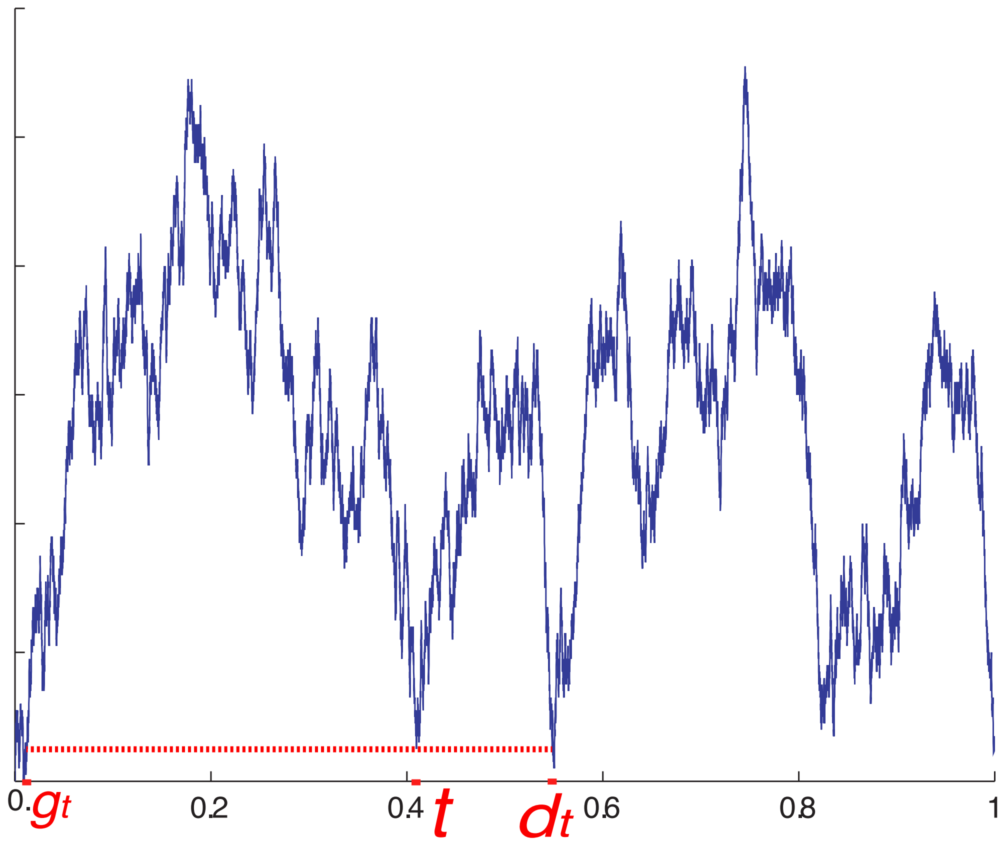

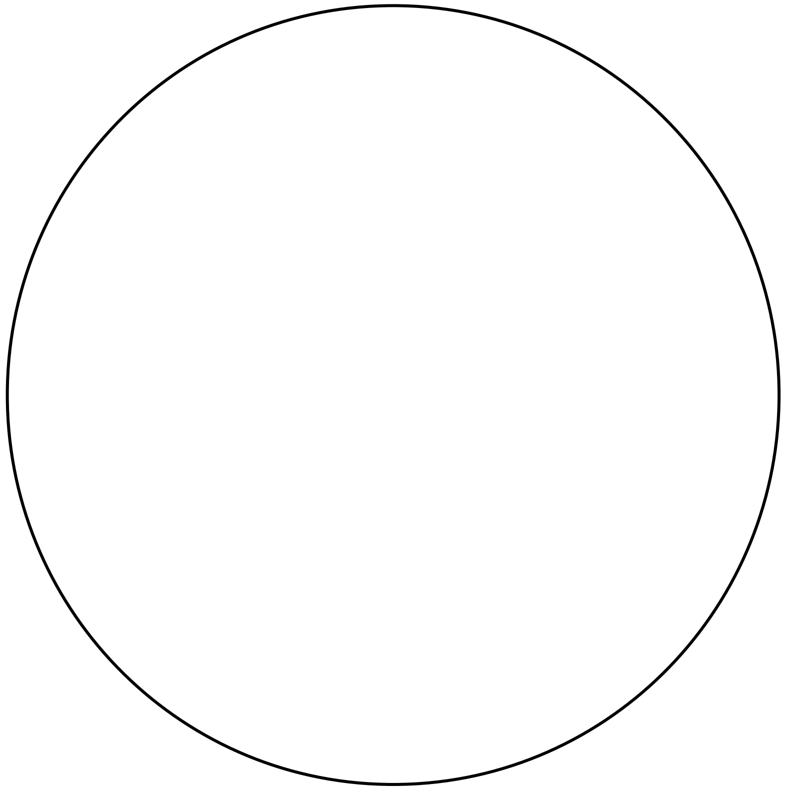

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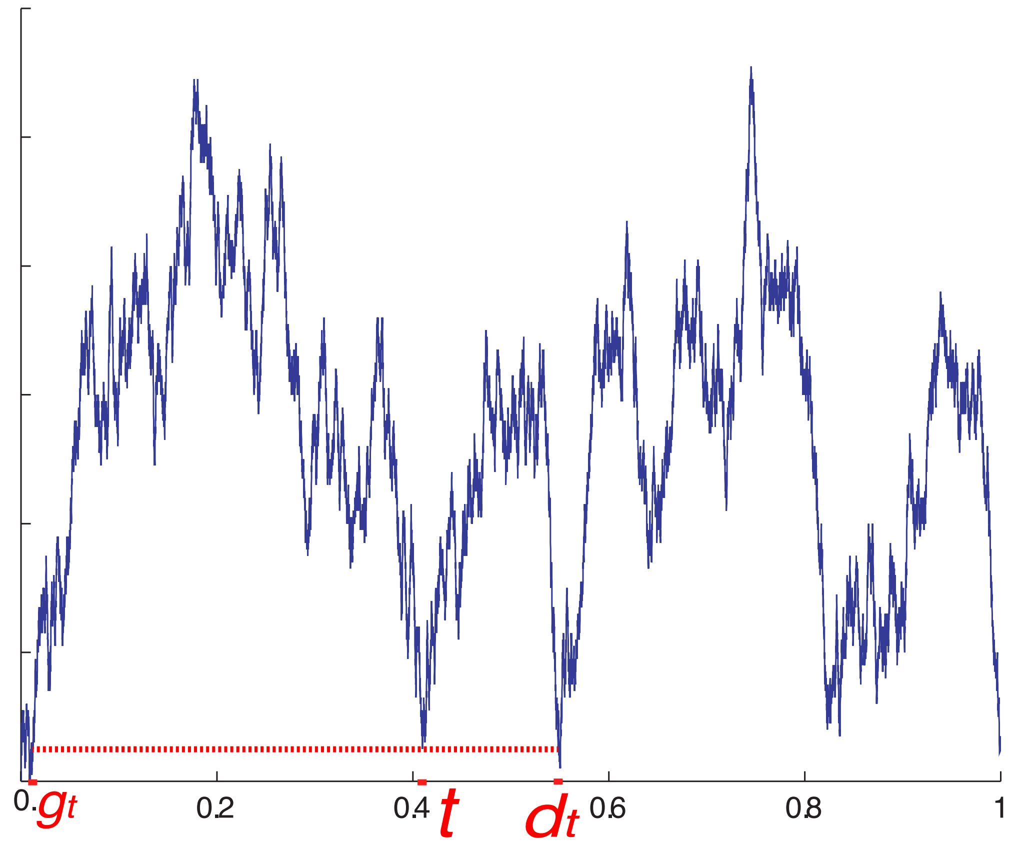

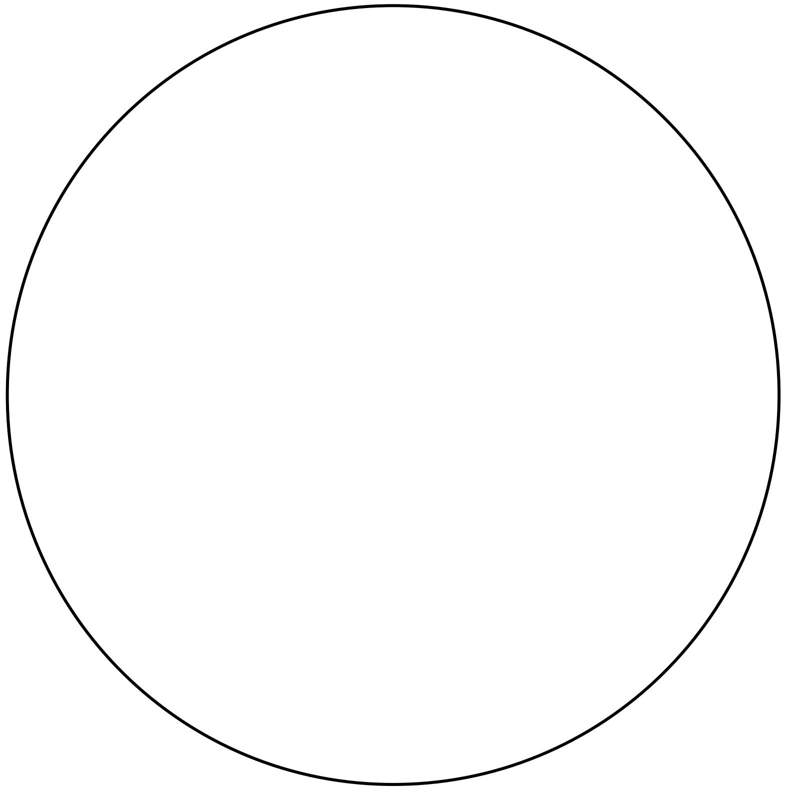

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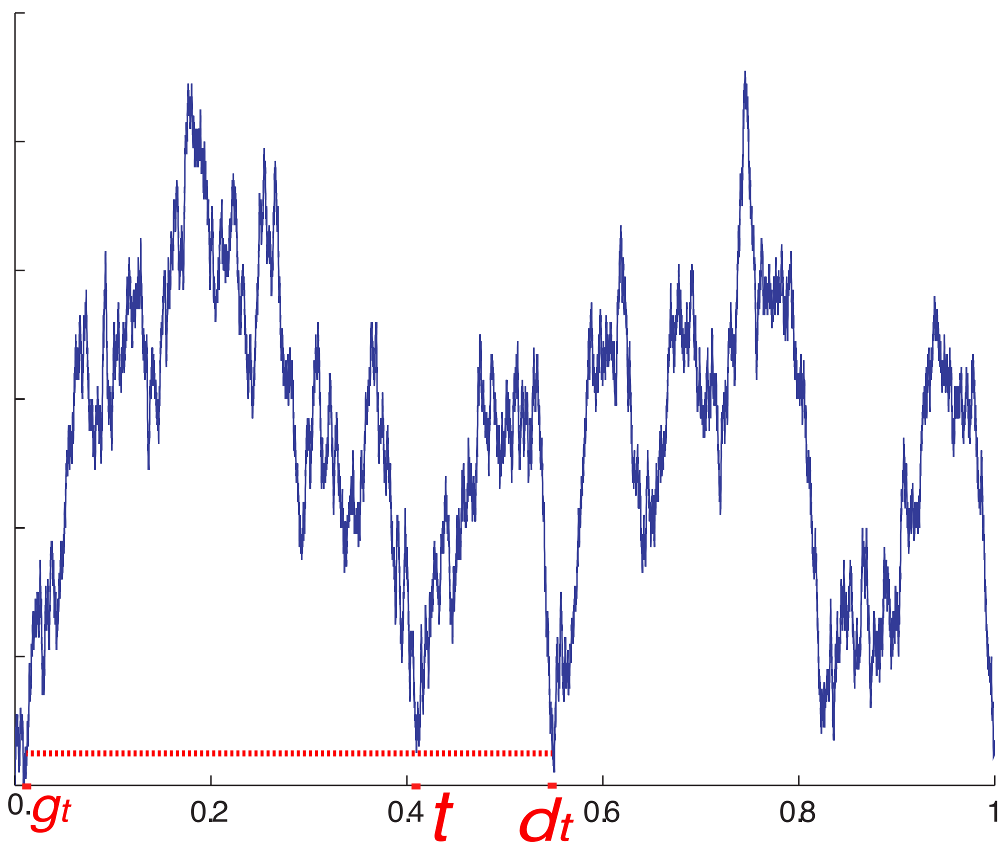

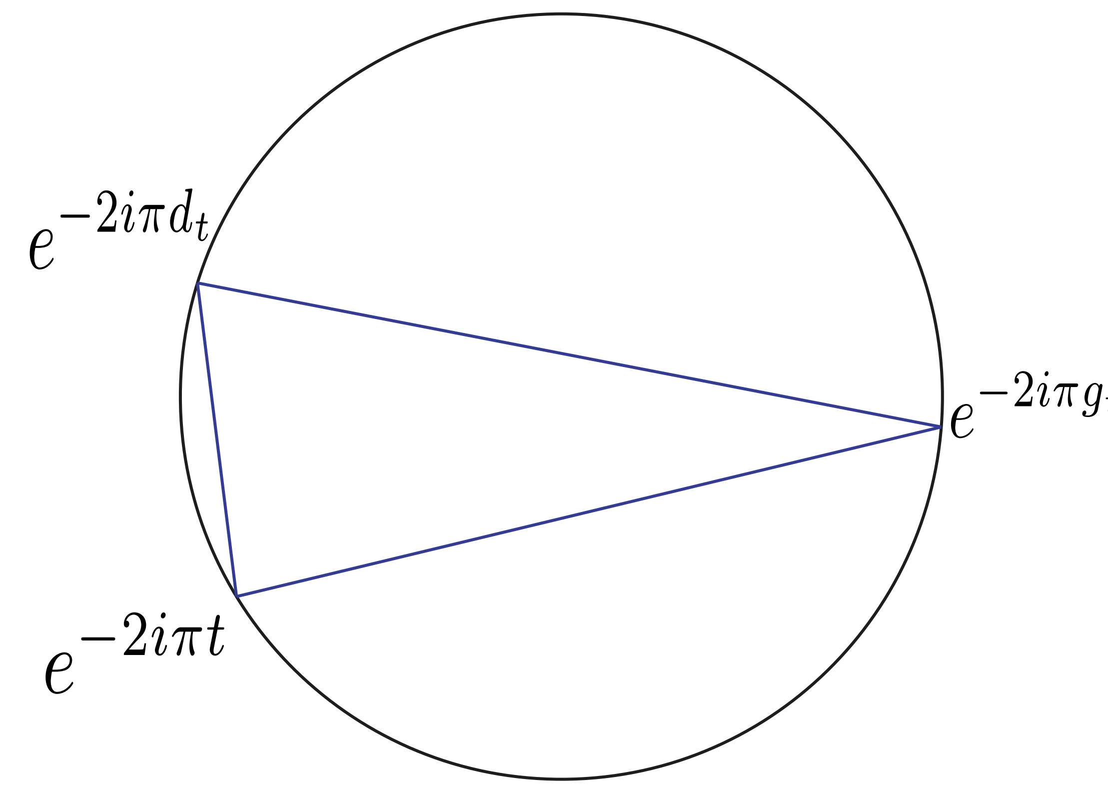

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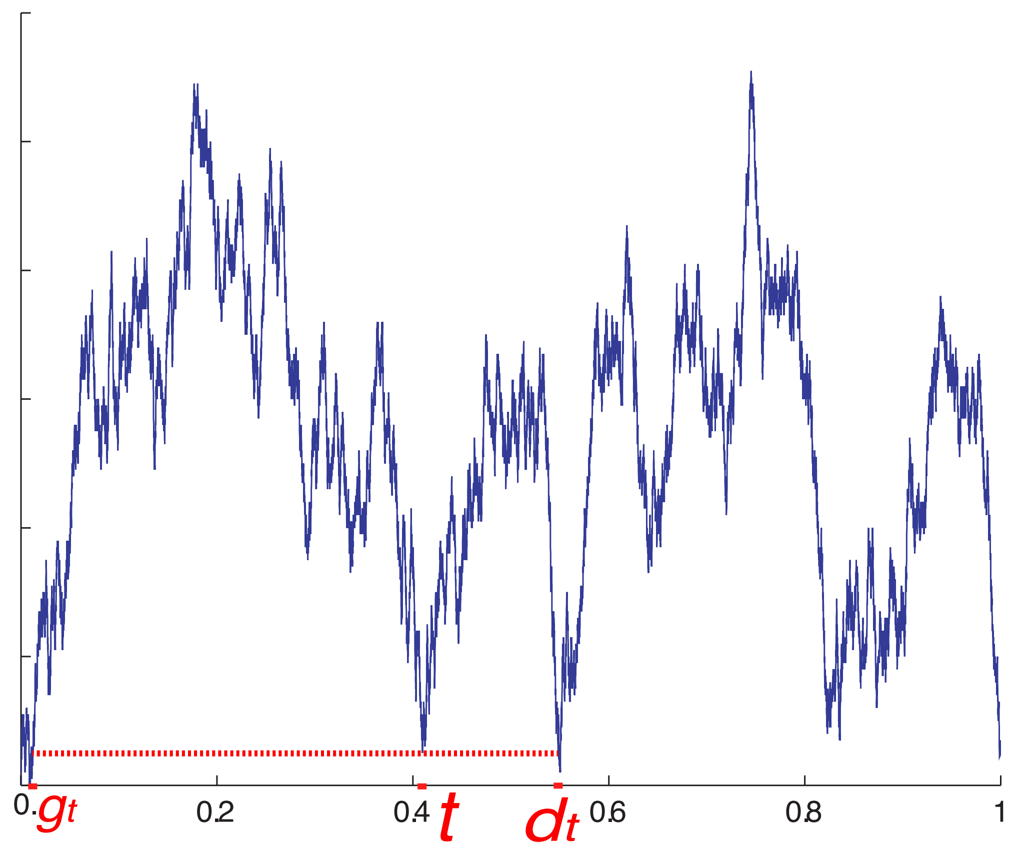

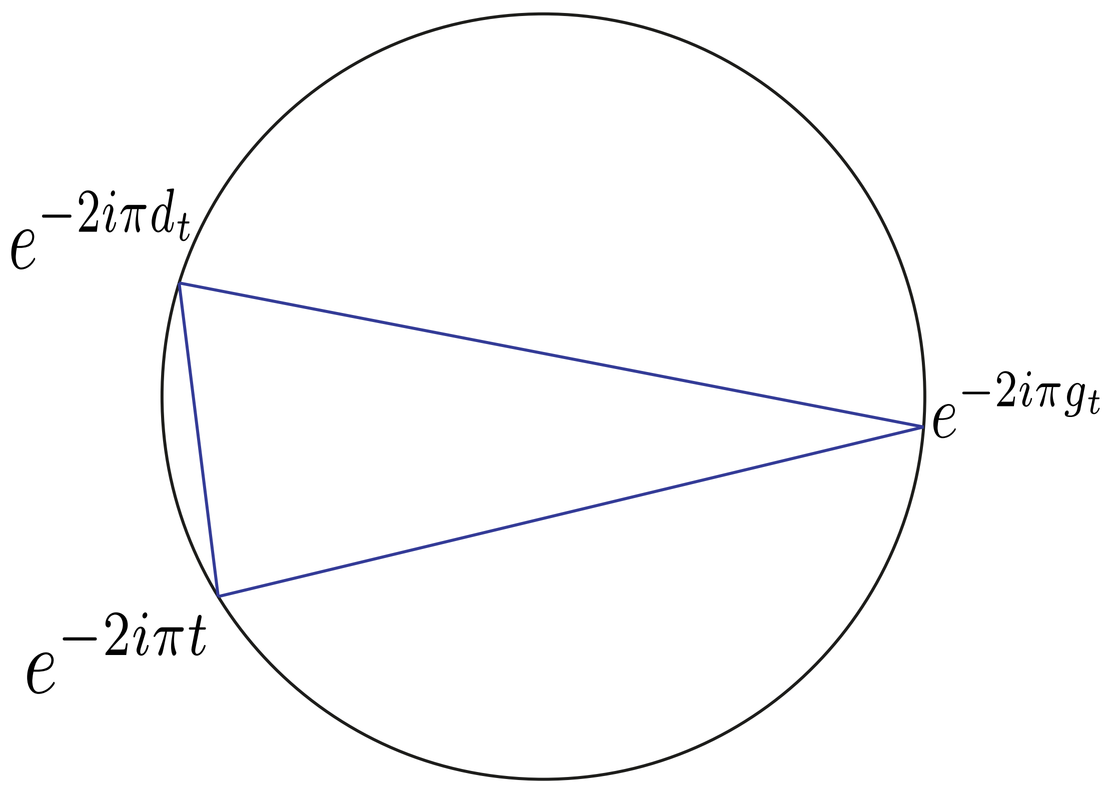

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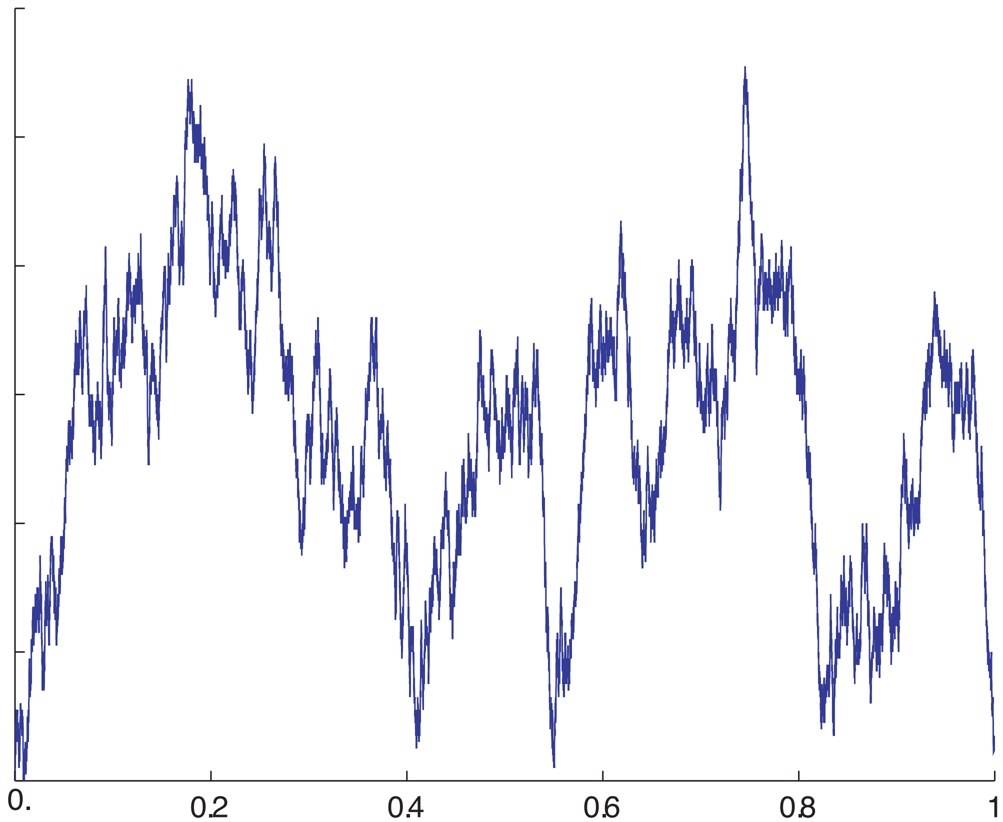

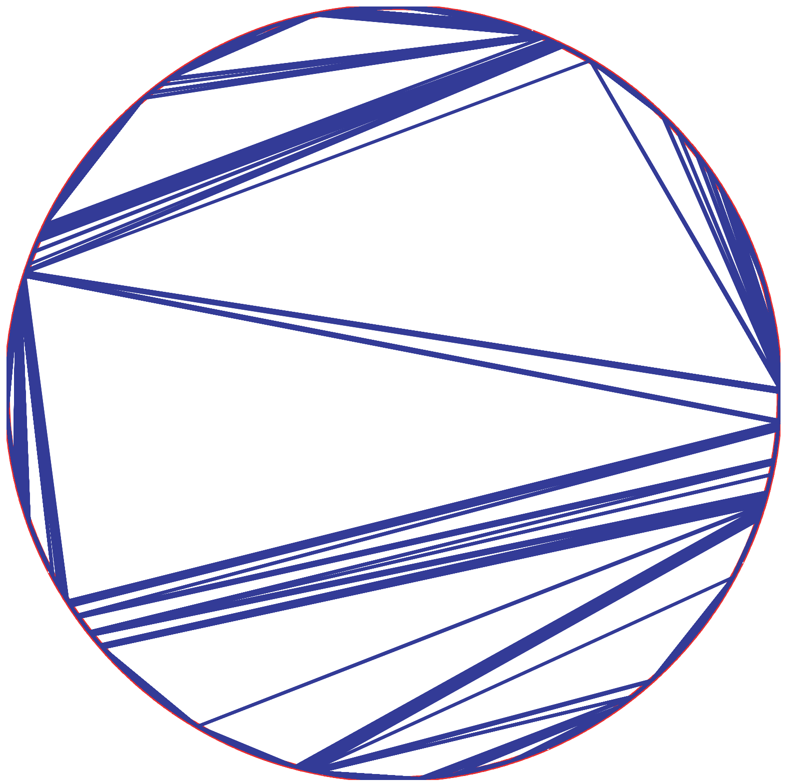

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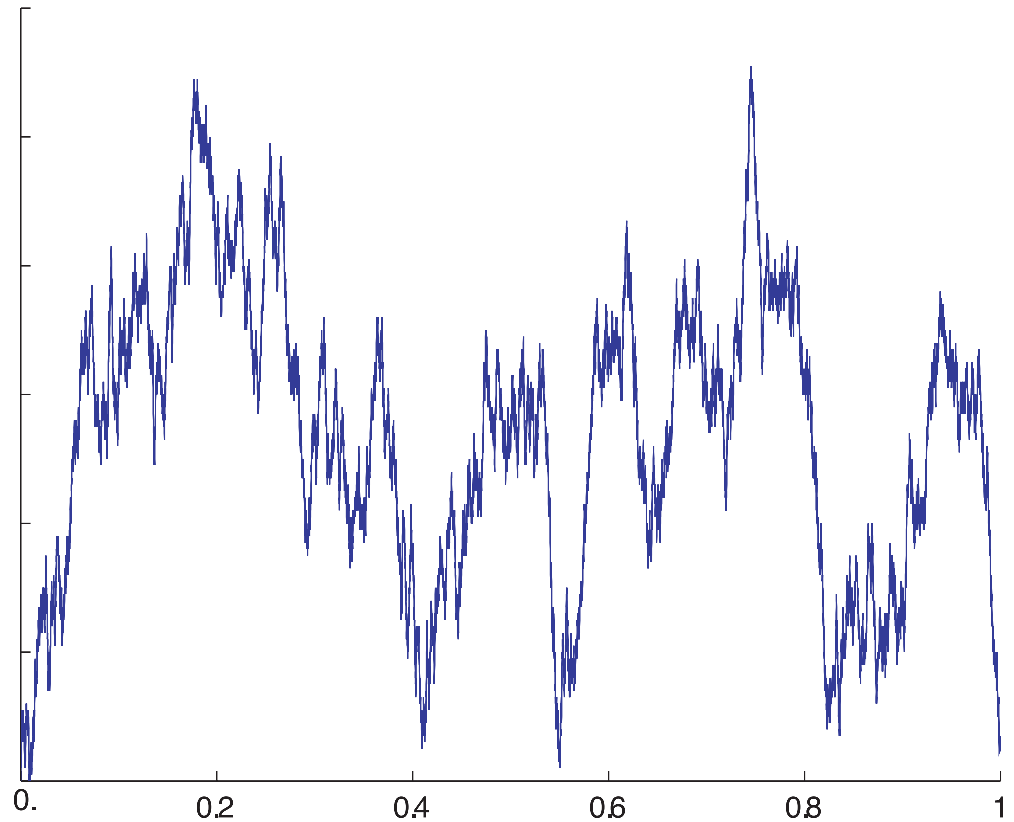

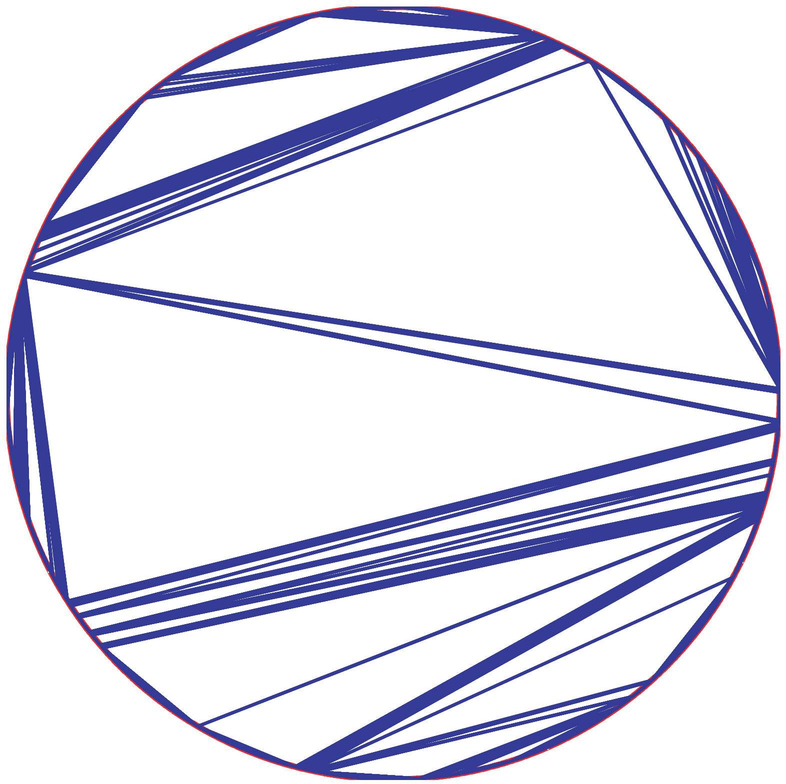

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The closure of this union, $L(\mathbb{e})$, is called the Brownian triangulation.

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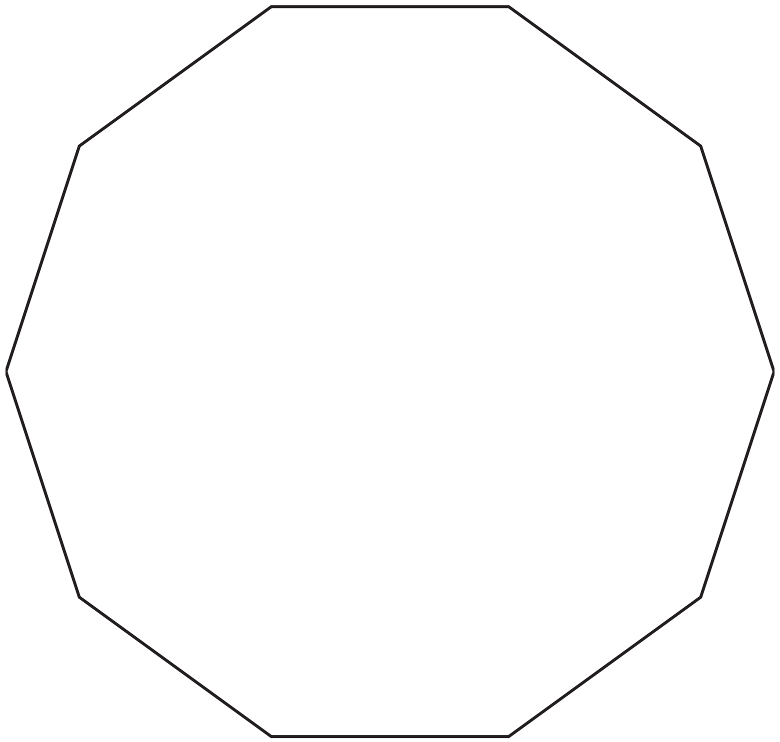

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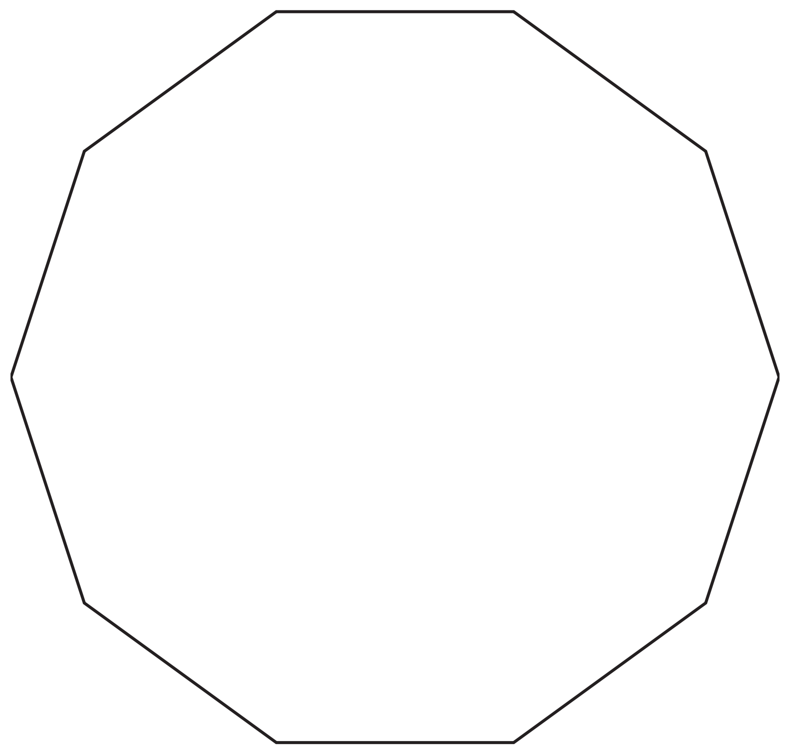

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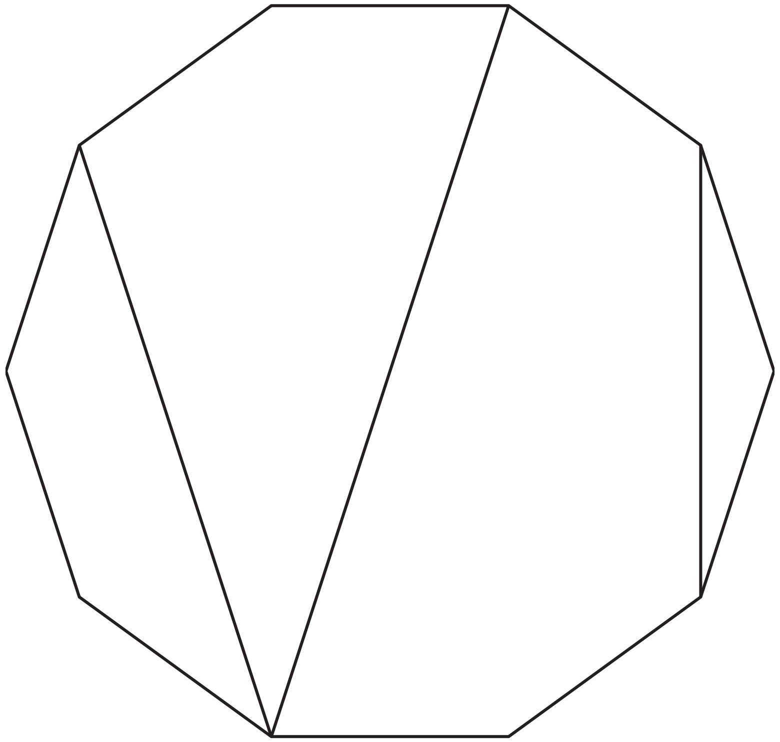

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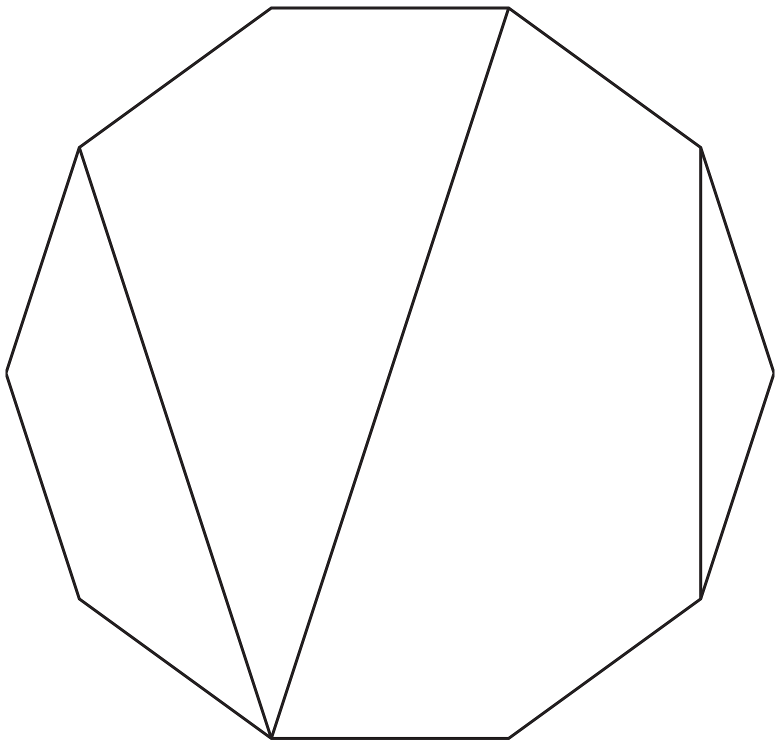

Figure: A dissection of a 10-gon.

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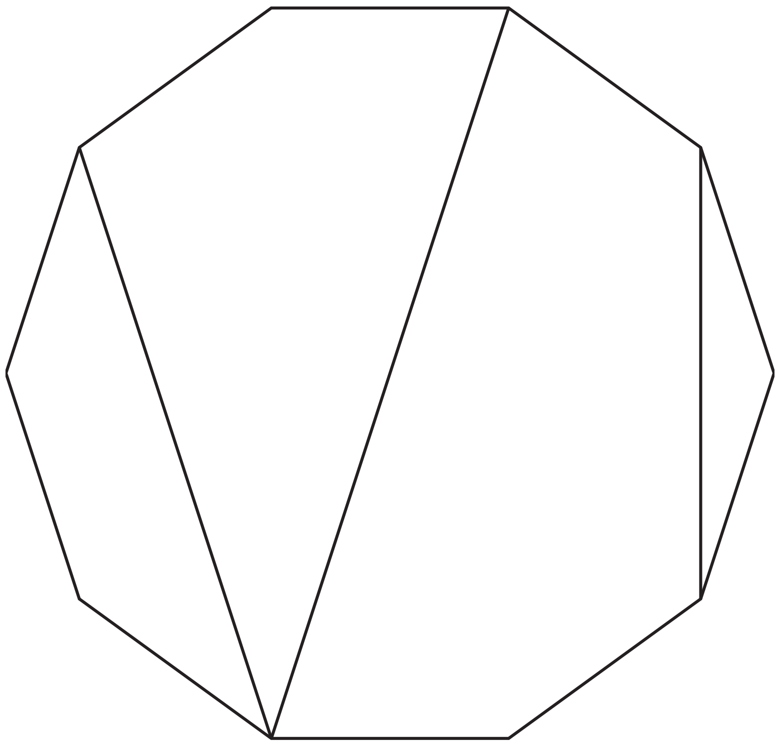

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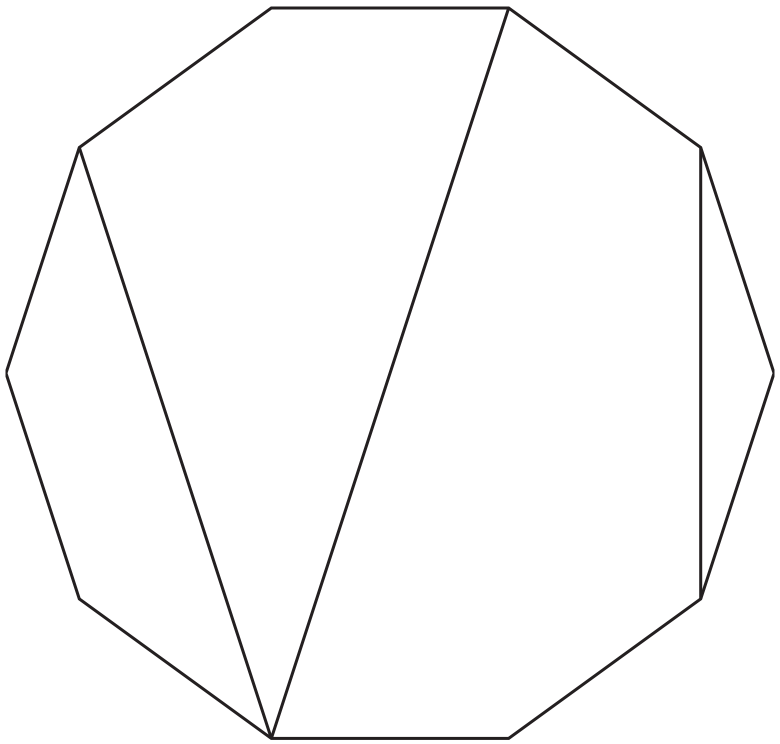

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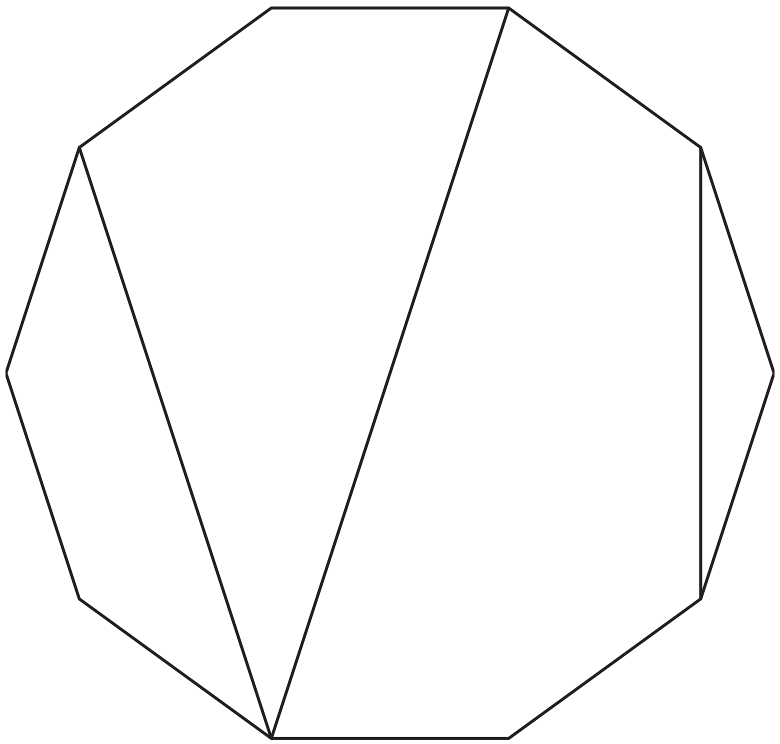

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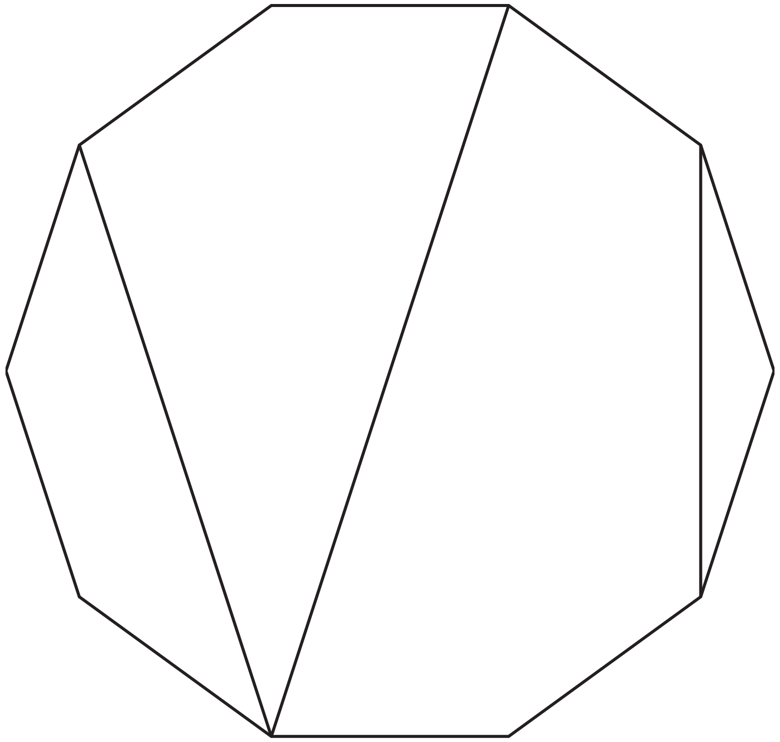

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$\leadsto$ Question: What does a large typical dissection look like?

Large typical dissections

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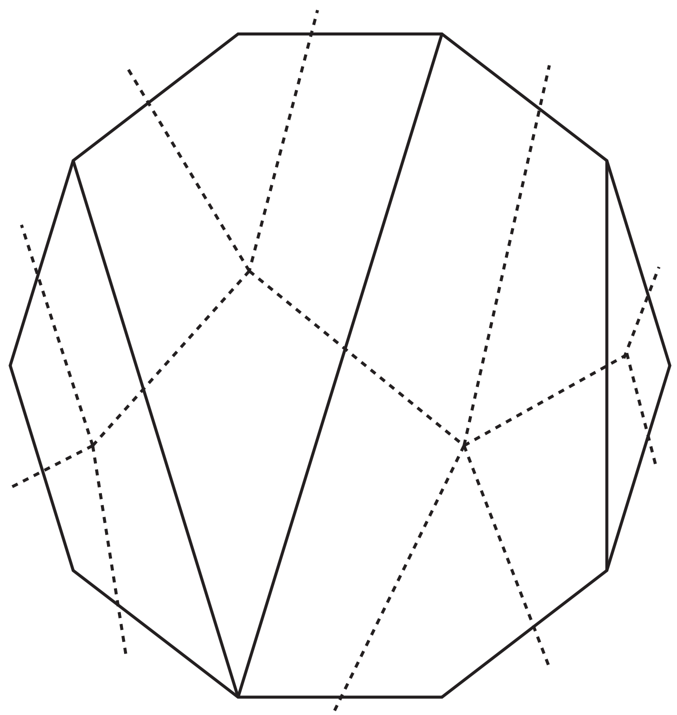

Figure: The dual tree of a dissection.

I. Trees

II. Triangulations

III. Minimal factorizations

Minimal factorizations

$\leadsto$ Question:

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$\#\mathfrak{M}_3 = 3$.

$\rightsquigarrow$ Question:

Minimal factorizations

Let $(1, 2, \ldots, n)$ be the n cycle.

Consider the set

$$\mathfrak{M}_n = \{(\tau_1, \ldots, \tau_{n-1}) \text{ transpositions } : \tau_1 \tau_2 \cdots \tau_{n-1} = (1, 2, \ldots, n)\}$$

of minimal factorizations (of the n-cycle into transpositions).

$\leadsto$ Question: $\#\mathfrak{M}_n = n^{n-2}$ (Dénes, 1959)

For example (multiply from left to right):

$$(1, 2, 3) = (1, 3)(2, 3) = (2, 3)(1, 2) = (1, 2)(1, 3),$$

$\#\mathfrak{M}_3 = 3.$

$\leadsto$ Question:

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$\#\mathfrak{M}_3 = 3$.

$\rightsquigarrow$ Question: for n large, what does a typical minimal factorization look like?

What space for minimal factorizations?

What space for minimal factorizations?

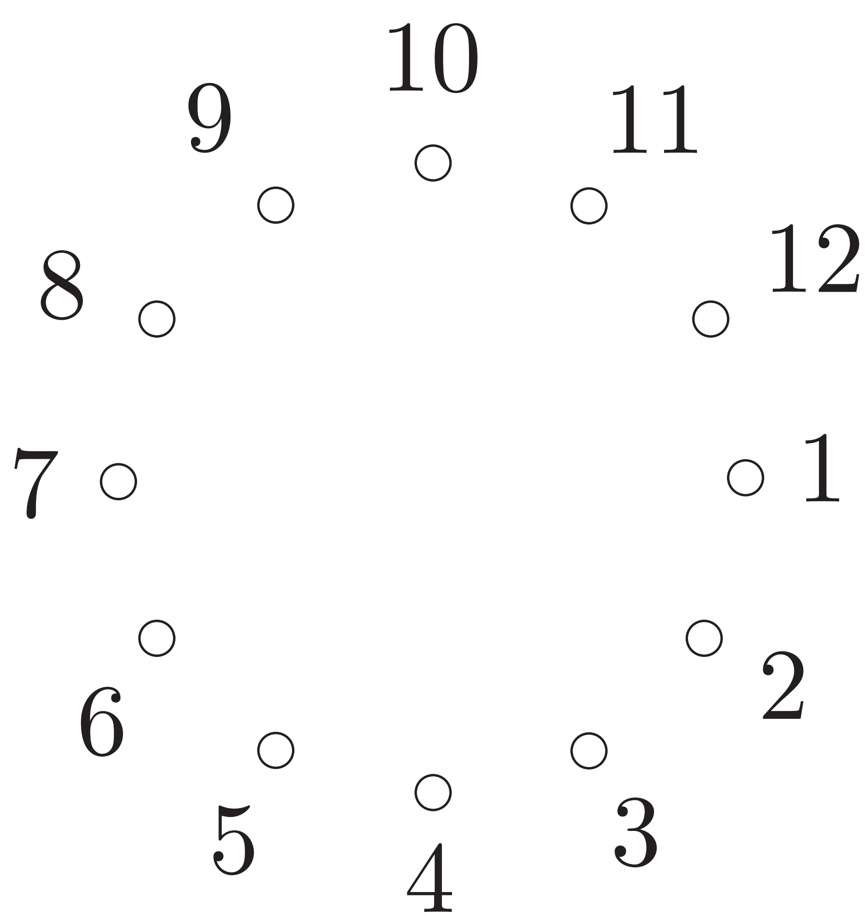
compact subsets of the unit disk.

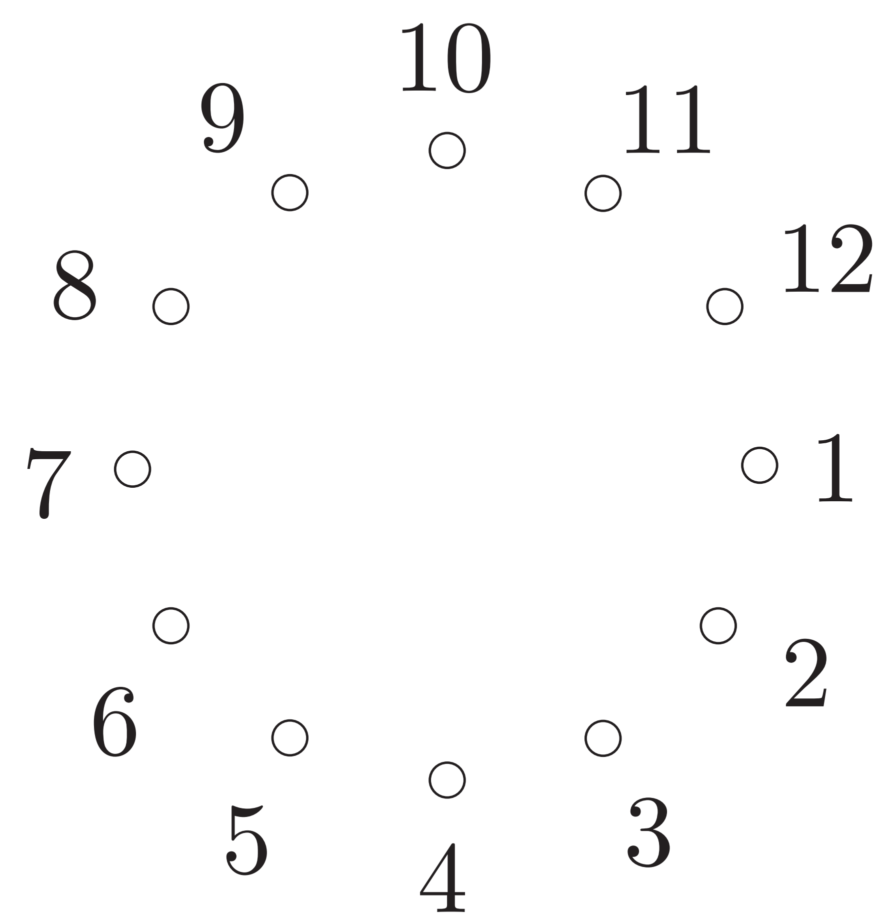
If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

Example ($n = 12$). Take

$$\big((1,3), (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11), (9,11)\big)$$





If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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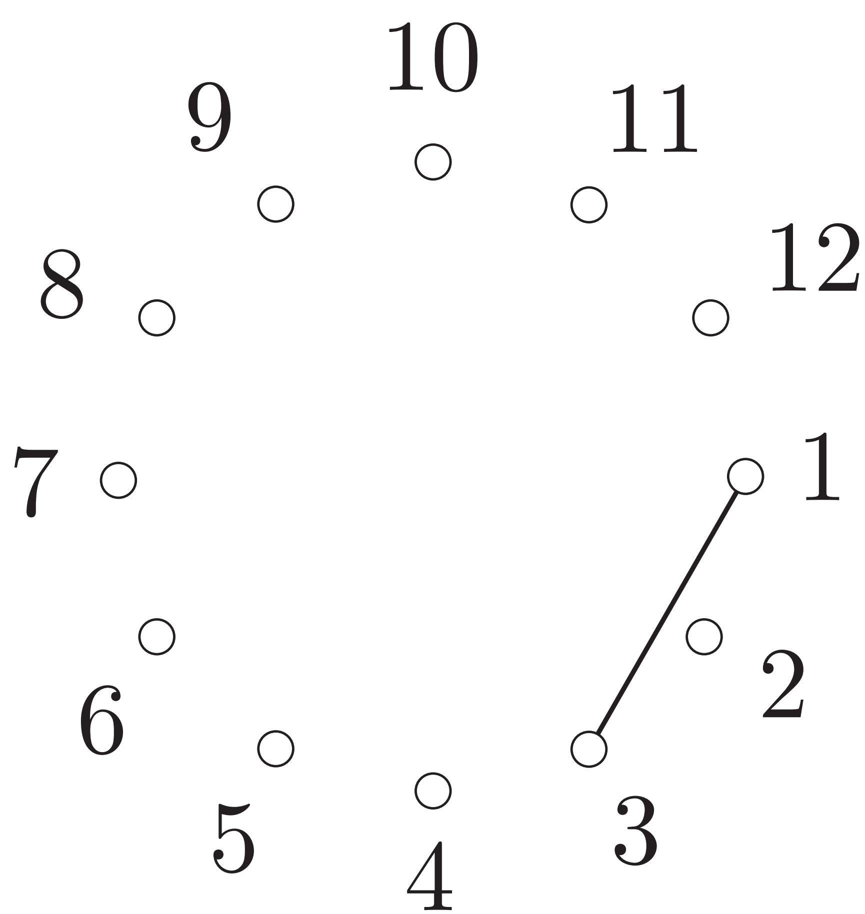
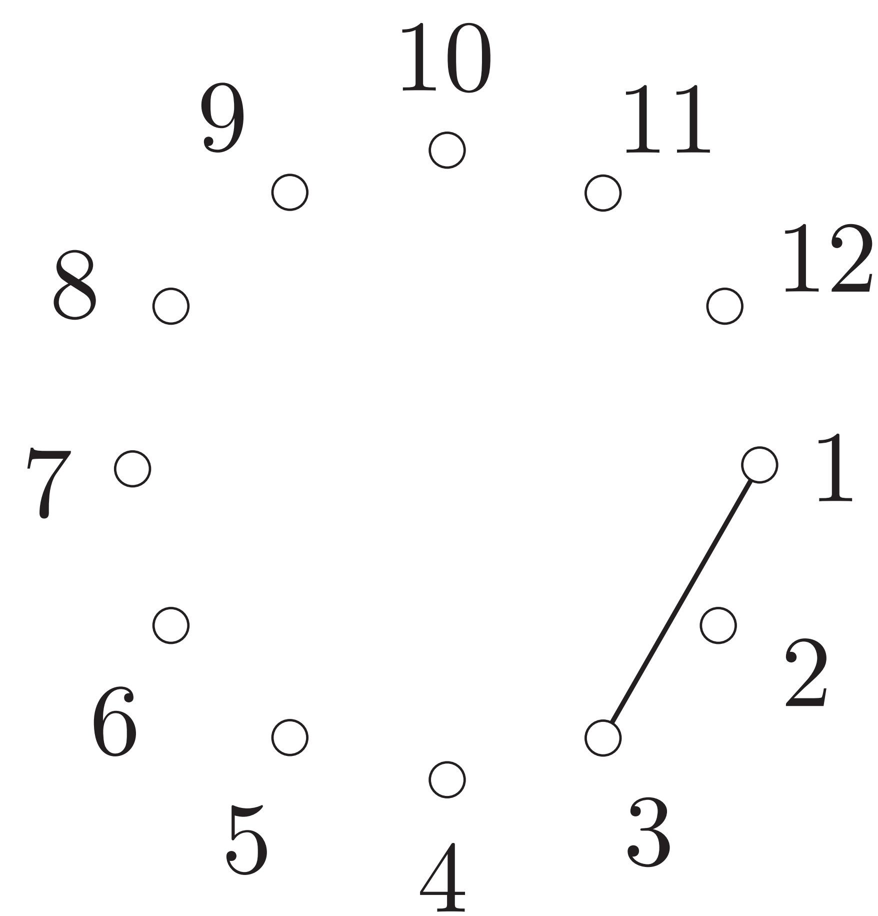
$\rightsquigarrow$ Example $(n = 12)$). For $k = 1$:

$$\Big(\underbrace{(1,3)}_{\text{product}=(1,3)}, (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11), (9,11$$



$\mathcal{F}_1$



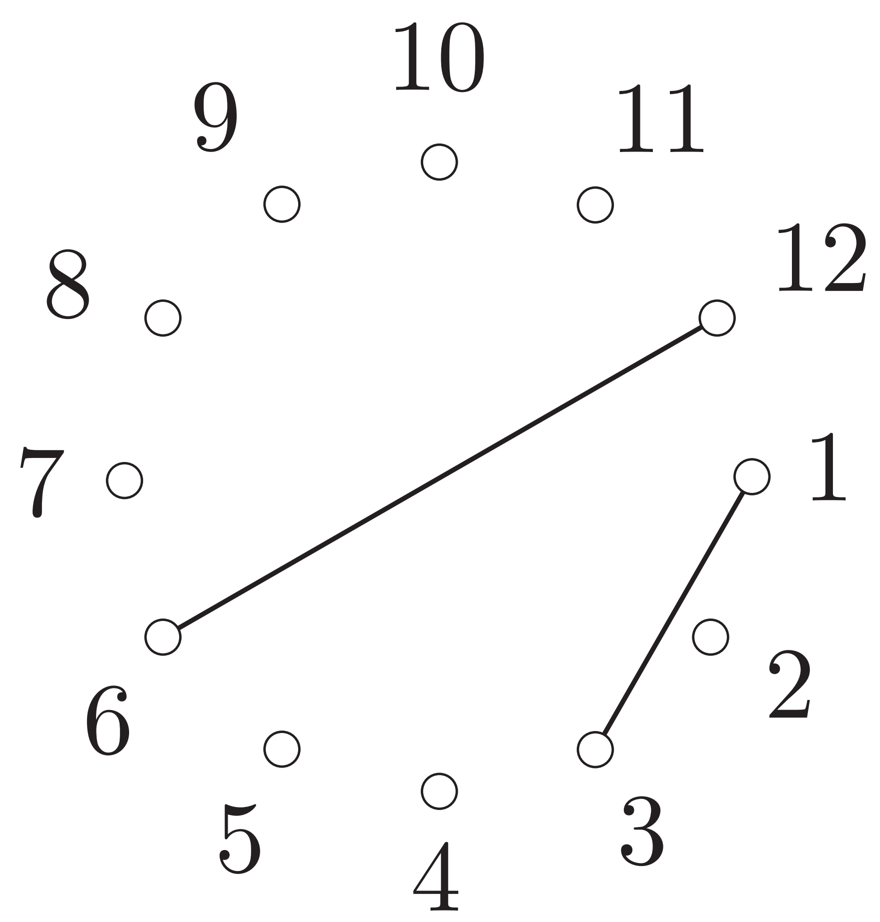
$\mathcal{P}_1$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

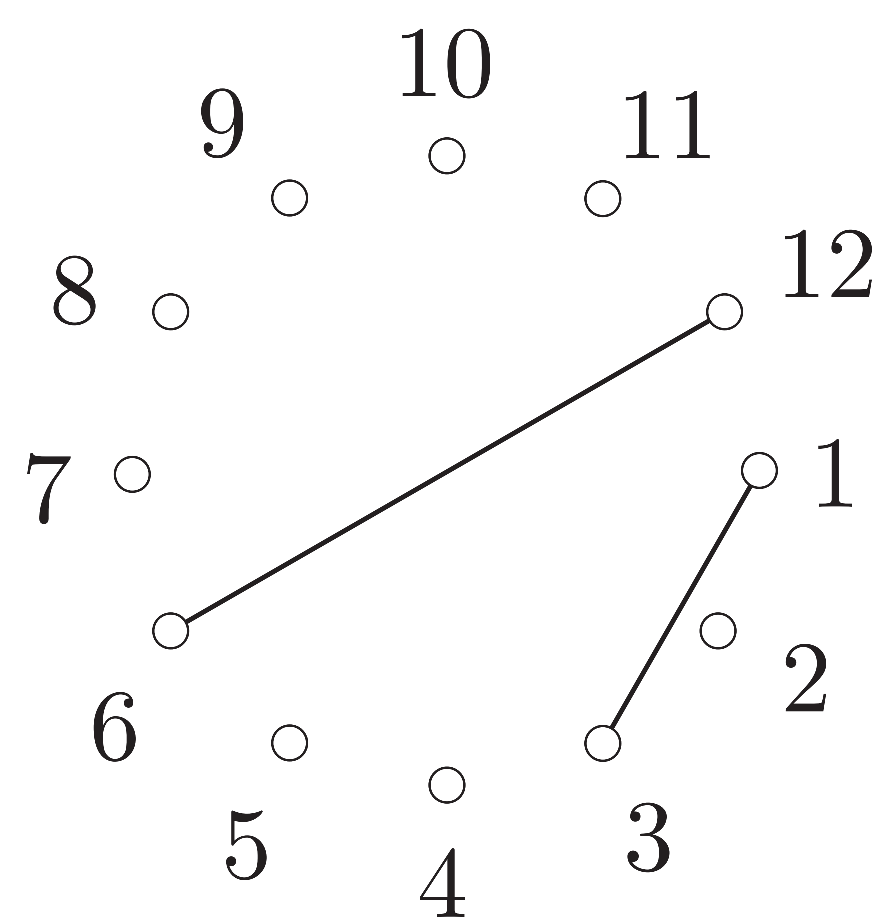
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1\tau_2 \cdots \tau_k$.

Example ($n = 12$). For $k = 2$:

$$\Big(\underbrace{(1,3),(6,12)}_{\text{product}=(1,3)(6,12)}, (1,5),(7,12),(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11) \Big)$$



$\mathcal{F}_2$



$\mathcal{P}_2$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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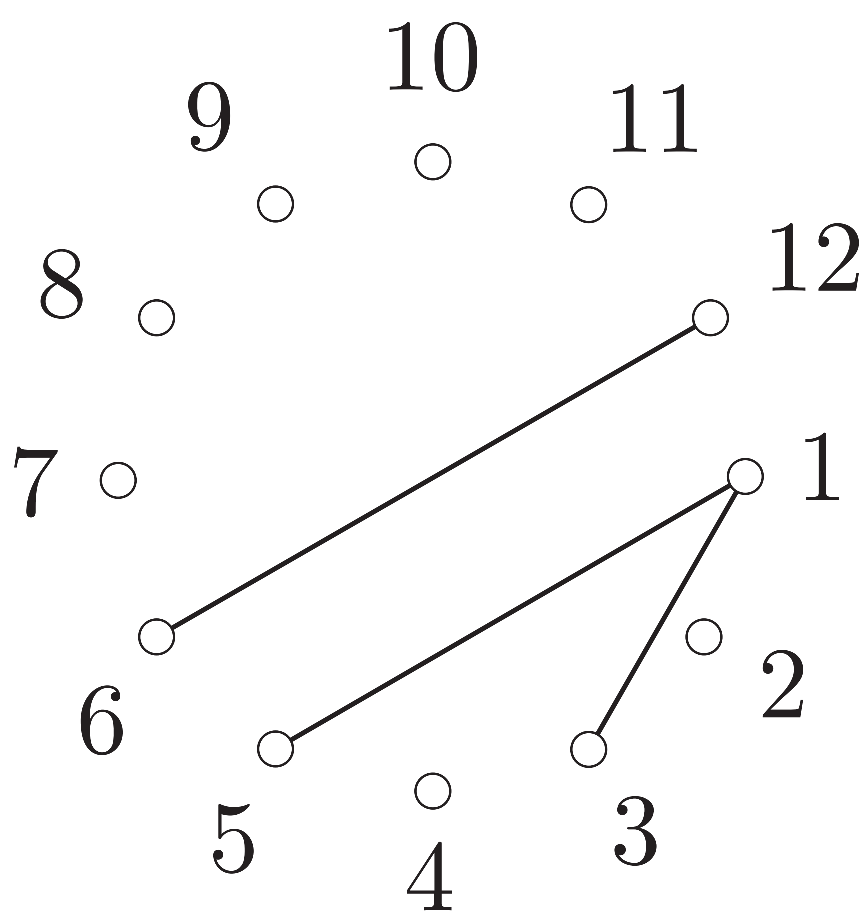
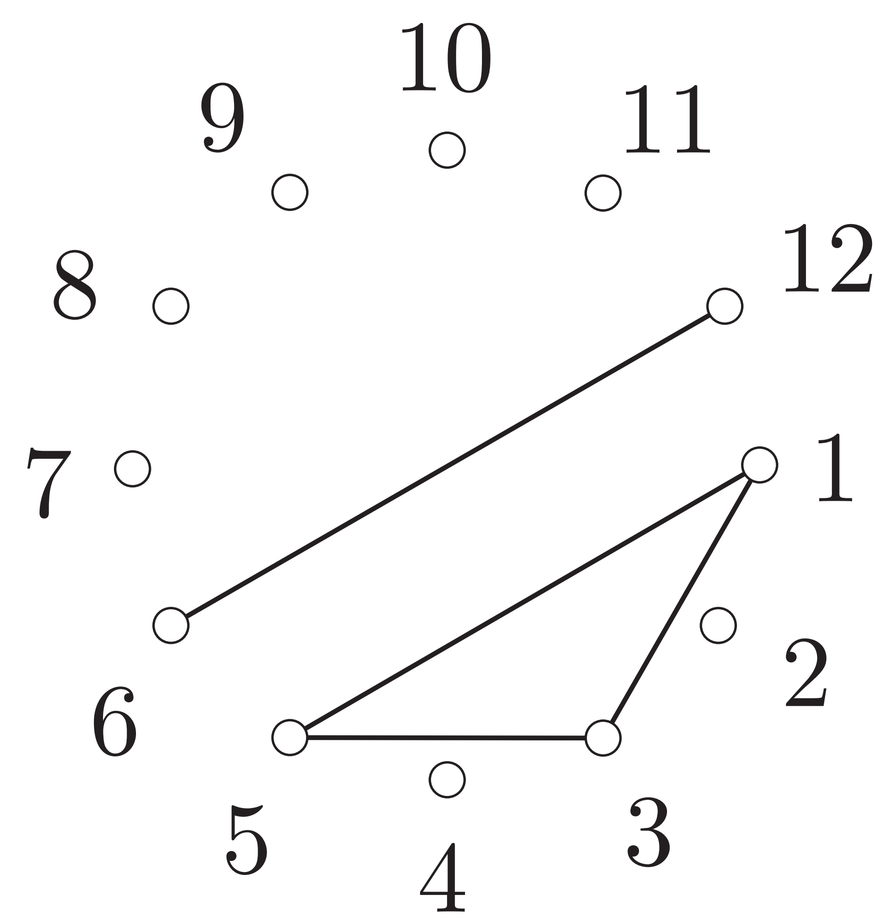
$\rightsquigarrow$ Example ($n = 12$)). For $k = 3$:

$$\Big(\underbrace{(1,3), (6,12), (1,5)}_{\text{product}=(1,3,5)(6,12)}, (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11), (9,11) \Big)$$



$\mathcal{F}_3$



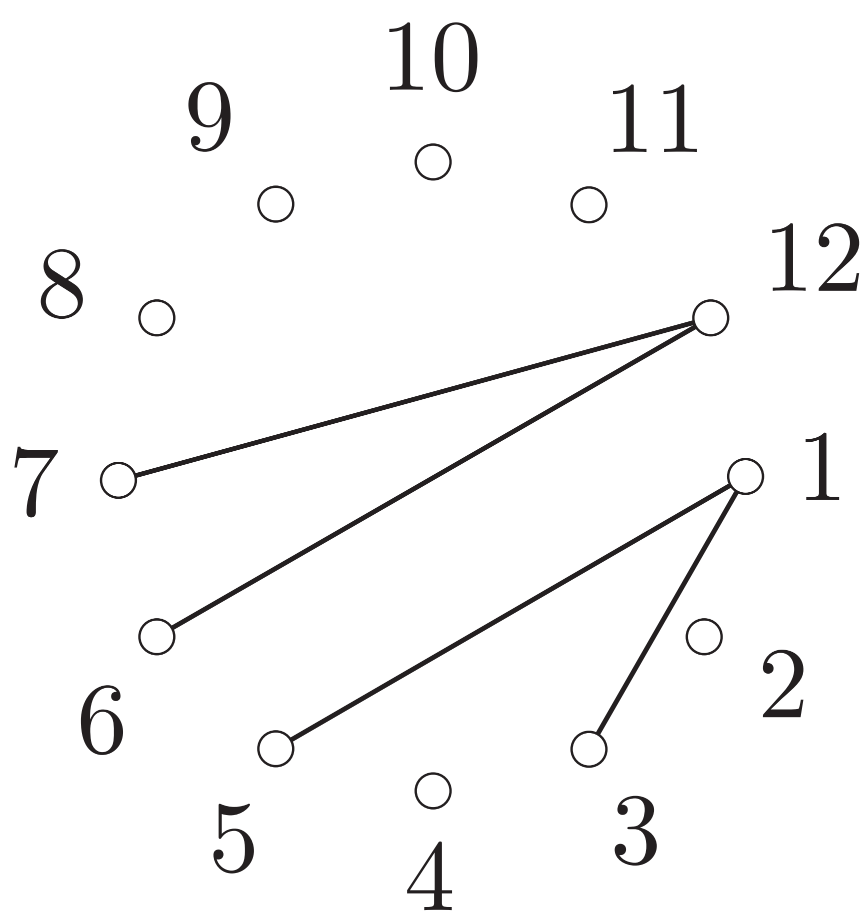
$\mathcal{P}_3$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

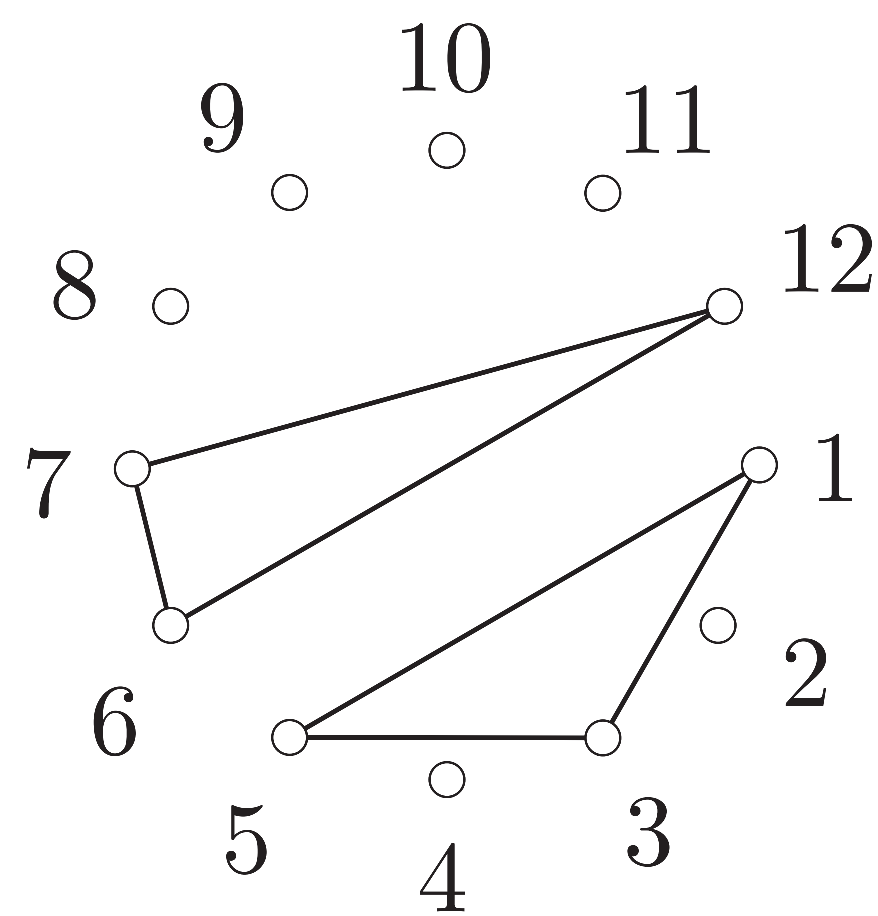
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1\tau_2\cdots\tau_k$.

$\leadsto$ Example ($n = 12$). For $k = 4$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12)}_{\text{product}=(1,3,5)(6,7,12)},(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11)\Big)$$



$\mathcal{F}_4$



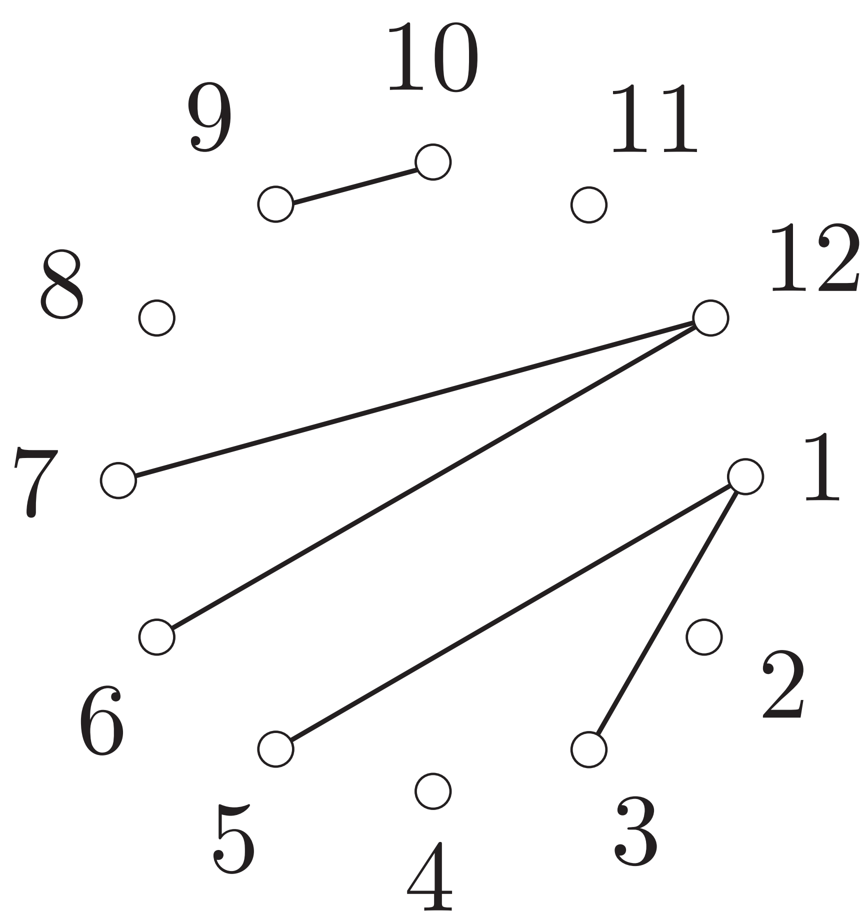
$\mathcal{P}_4$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

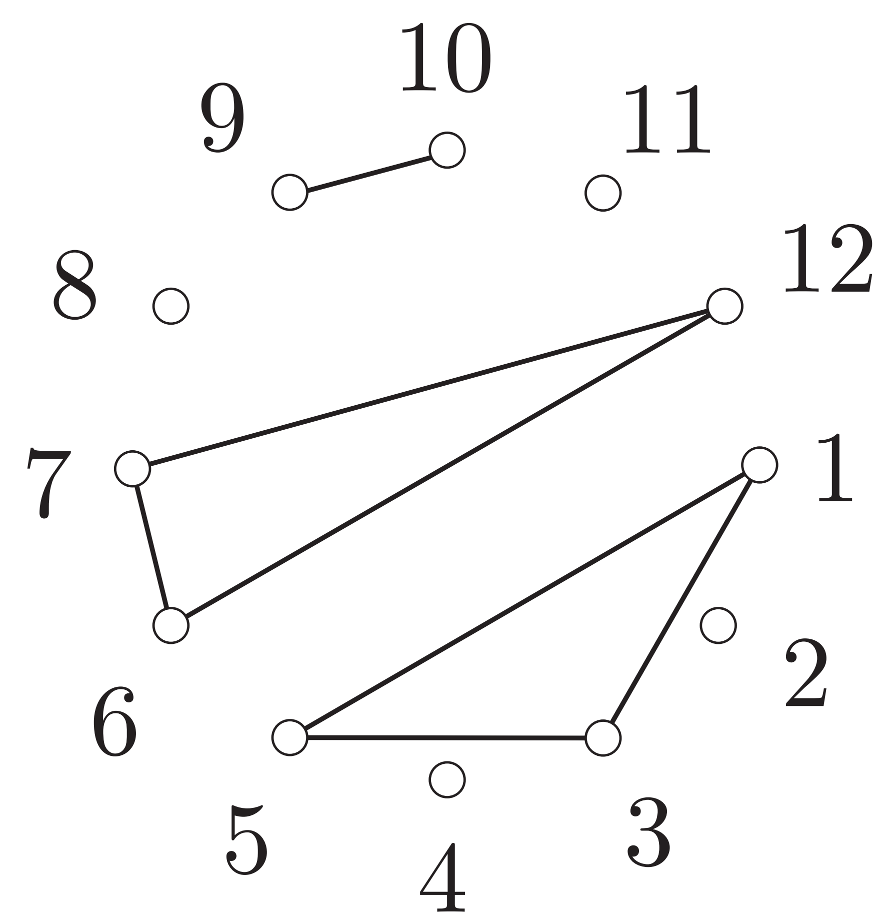
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

⇝ Example ($n = 12$). For $k = 5$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10)}_{\text{product}=(1,3,5)(6,7,12)(9,10)},(11,12),(2,3),(4,5),(1,6),(8,11),(9,11) \Big)$$



$\mathcal{F}_5$



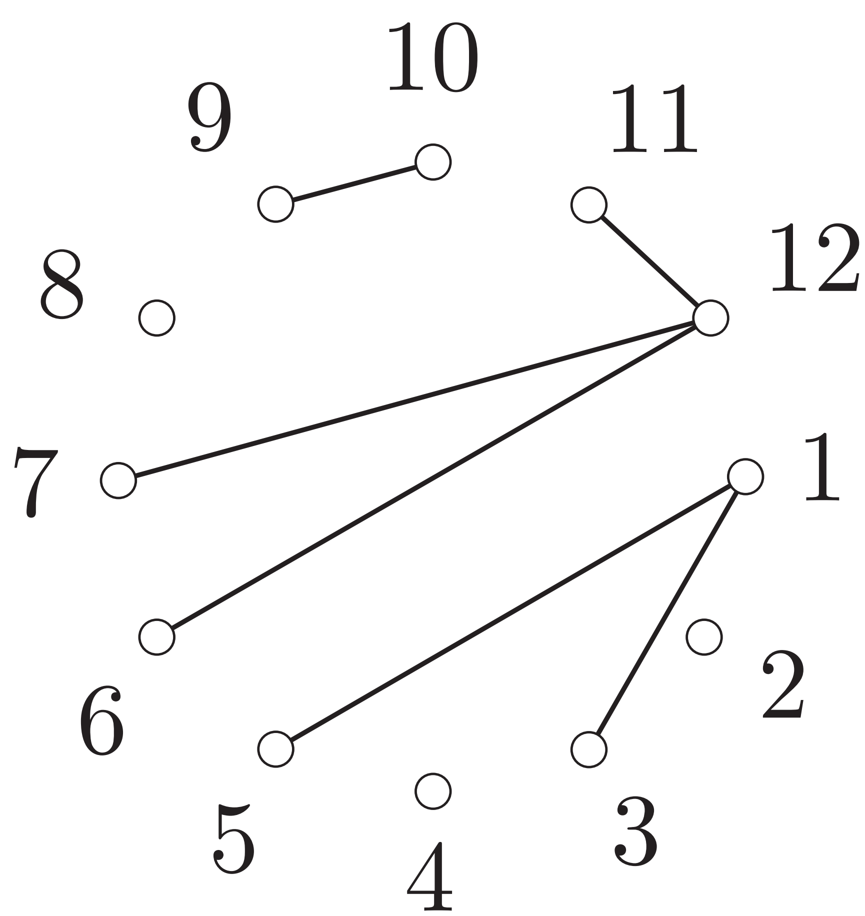
$\mathcal{P}_5$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

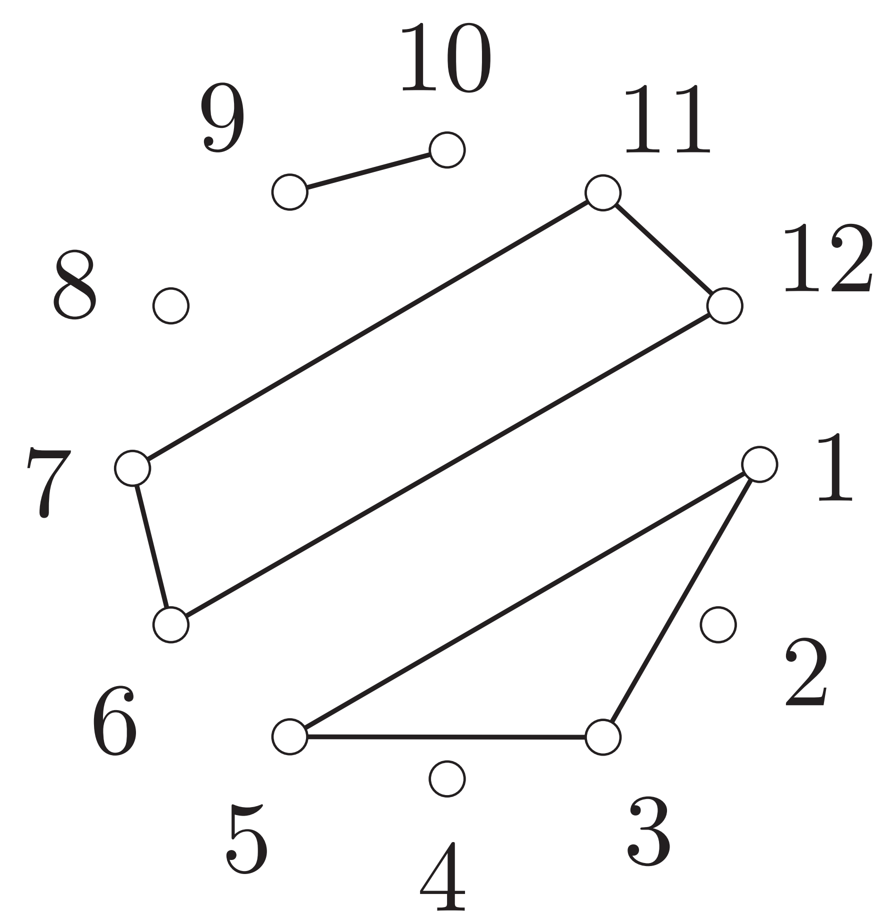
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

$\rightsquigarrow$ Example ($n = 12$)). For $k = 6$:

$$\Big(\underbrace{(1,3), (6,12), (1,5), (7,12), (9,10), (11,12)}_{\text{product}=(1,3,5)(6,7,11,12)(9,10)}, (2,3), (4,5), (1,6), (8,11), (9,11) \Big)$$



$\mathcal{F}_6$



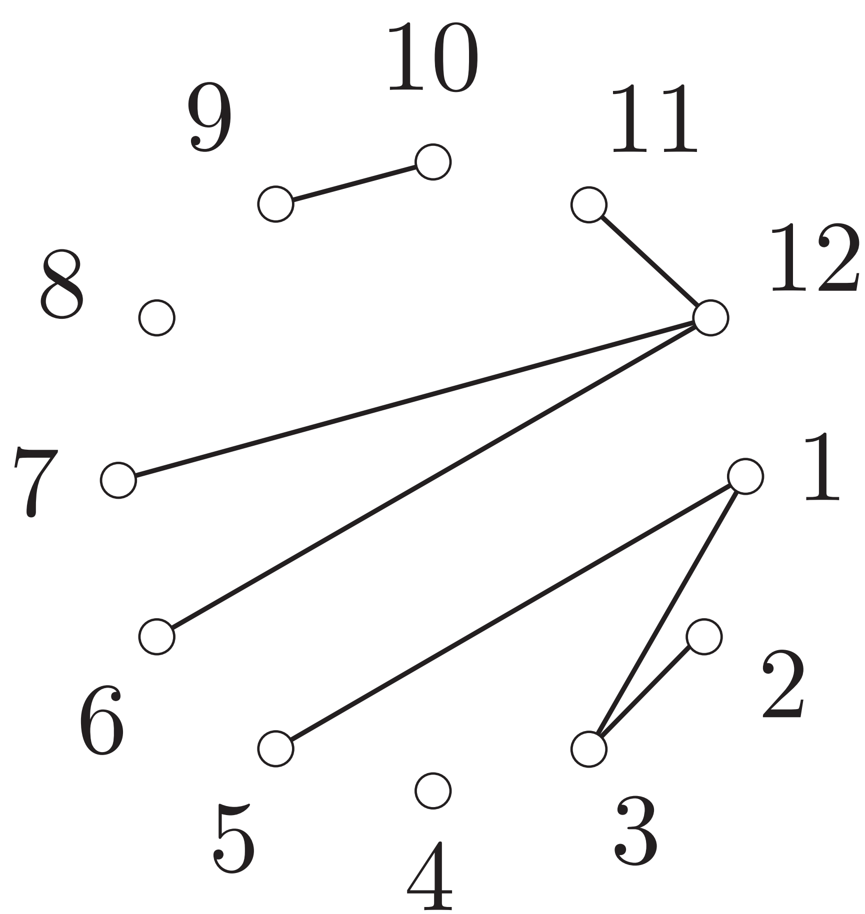
$\mathcal{P}_6$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

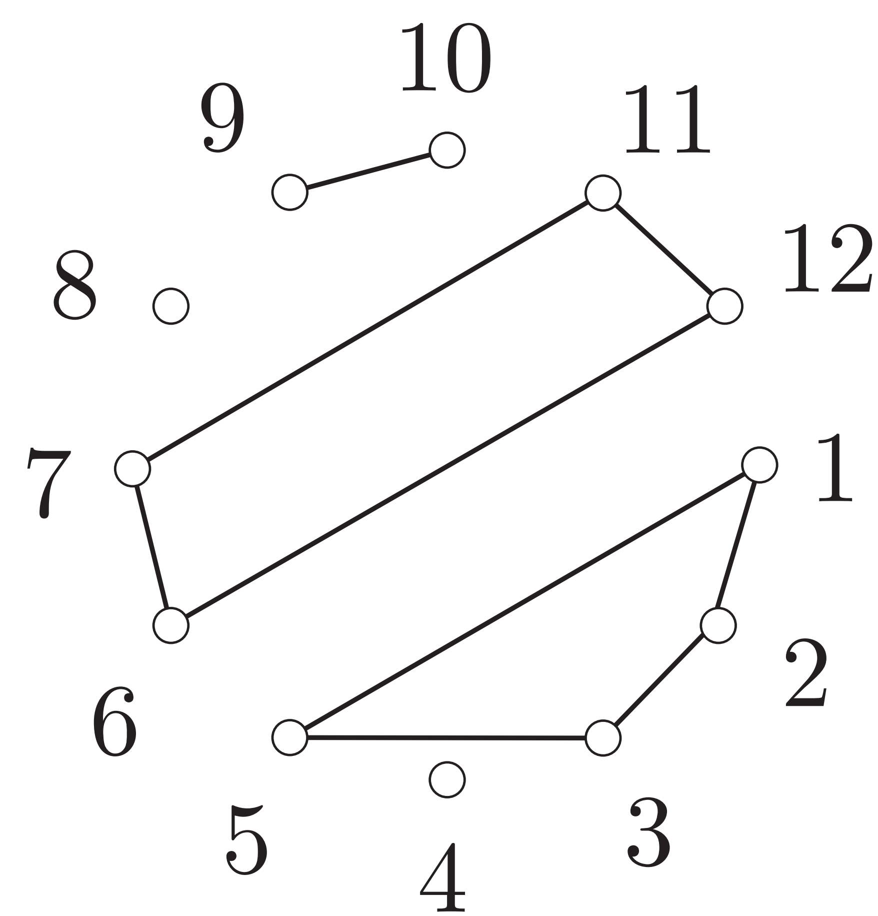
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

Example ($n = 12$)). For $k = 7$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3)}_{\text{product}=(1,2,3,5)(6,7,11,12)(9,10)}, (4,5),(1,6),(8,11),(9,11) \Big)$$



$\mathcal{F}_7$



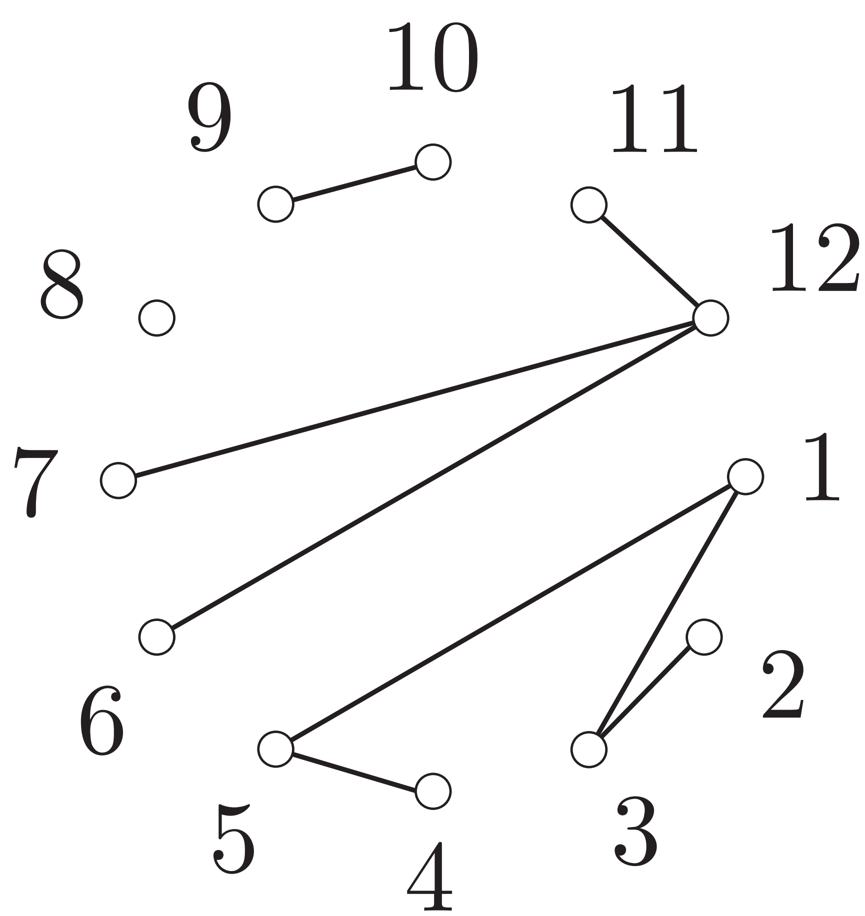
$\mathcal{P}_7$

If $(\tau_1, \dots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

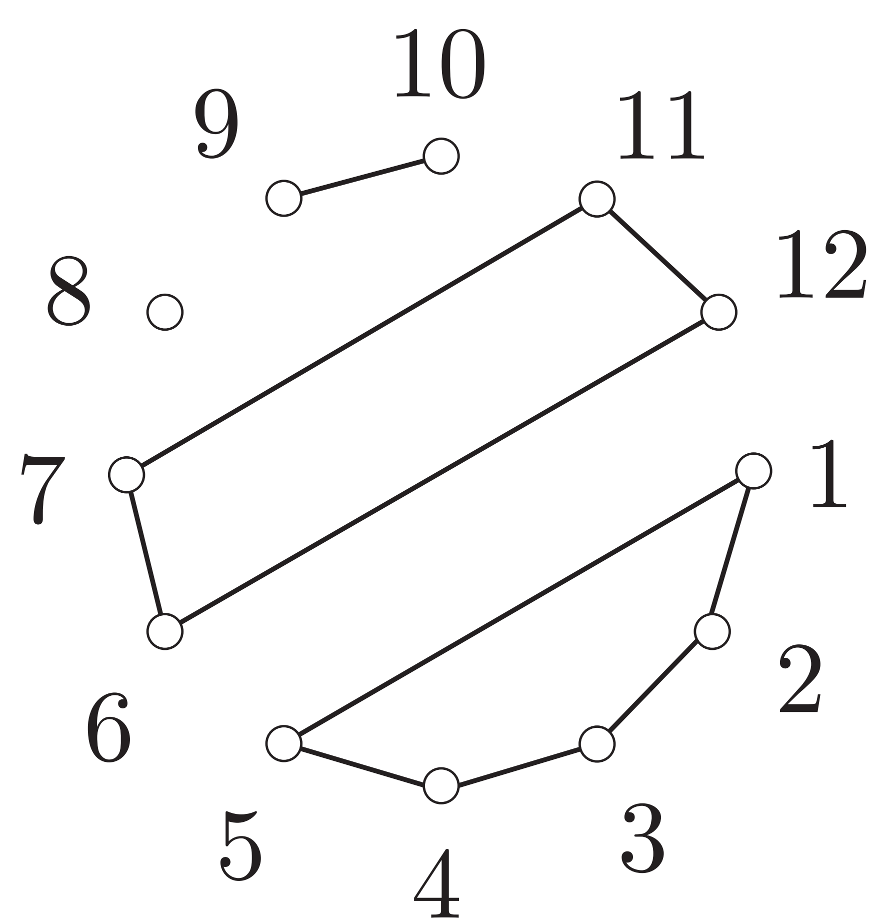
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

Example ($n = 12$)). For $k = 8$:

$$\Big(\underbrace{(1,3), (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5)}_{\text{product}=(1,2,3,4,5)(6,7,11,12)(9,10)}, (1,6), (8,11), (9,11) \Big)$$



$\mathcal{F}_8$



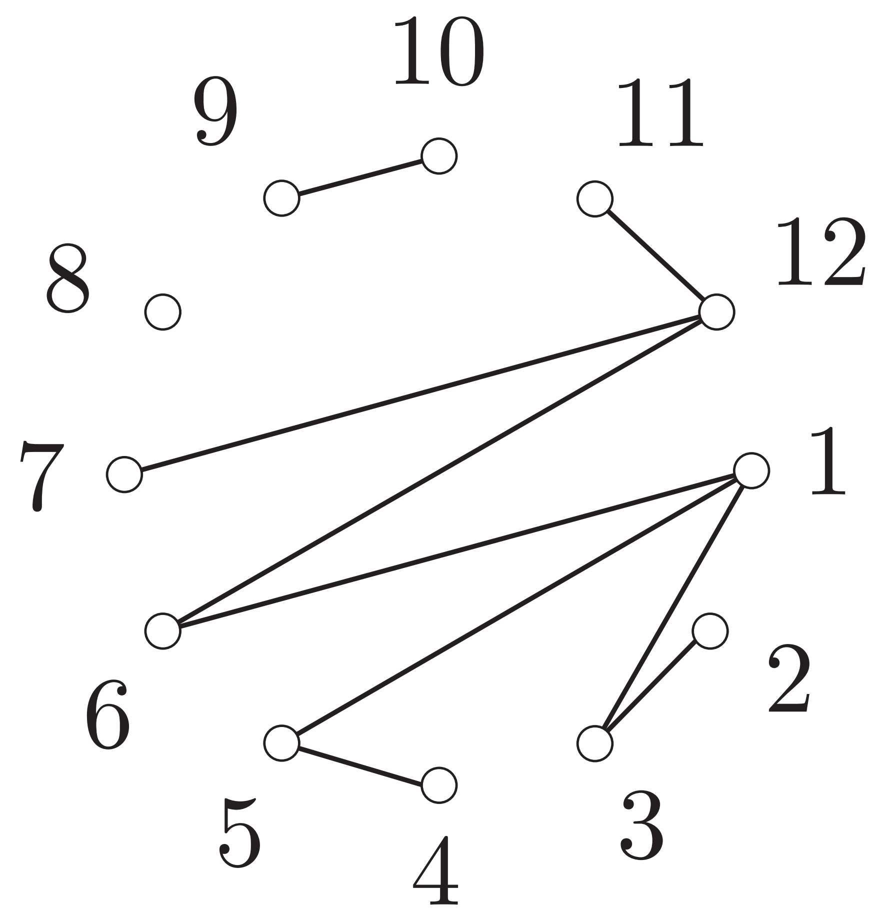
$\mathcal{P}_8$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

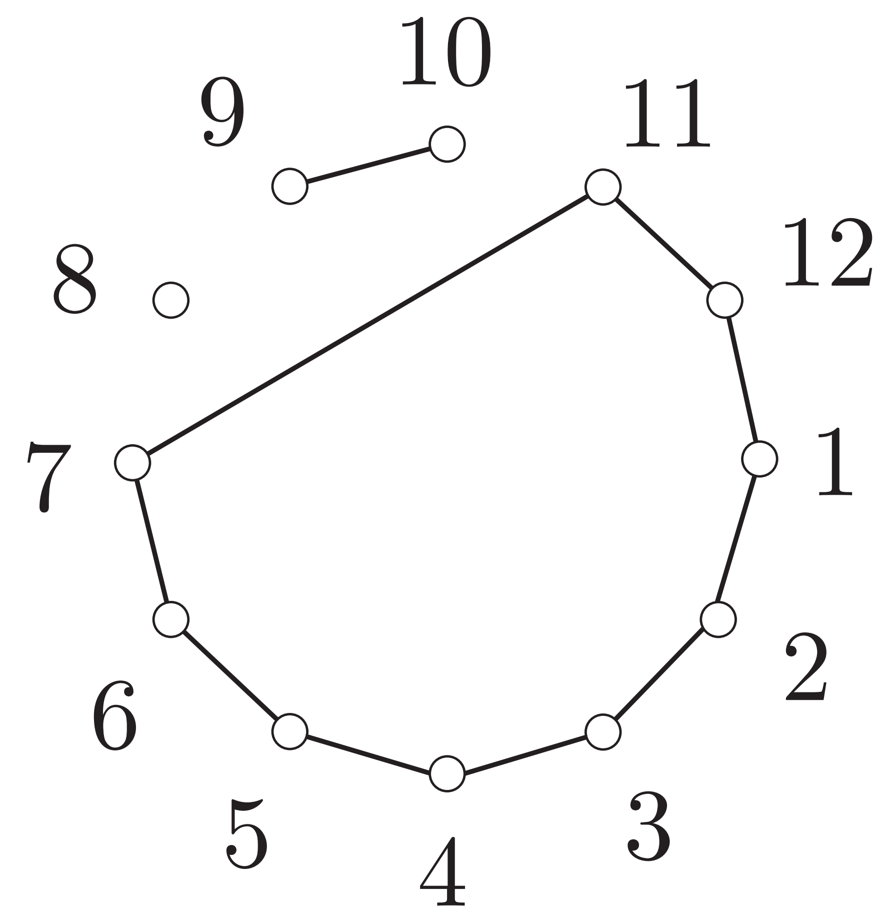
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

Example $(n = 12)$). For $k = 9$:

$$\Big(\underbrace{(1,3), (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6)}_{\text{product}=(1,2,3,4,5,6,7,11,12)(9,10)}, (8,11), (9,11) \Big)$$



$\mathcal{F}_9$



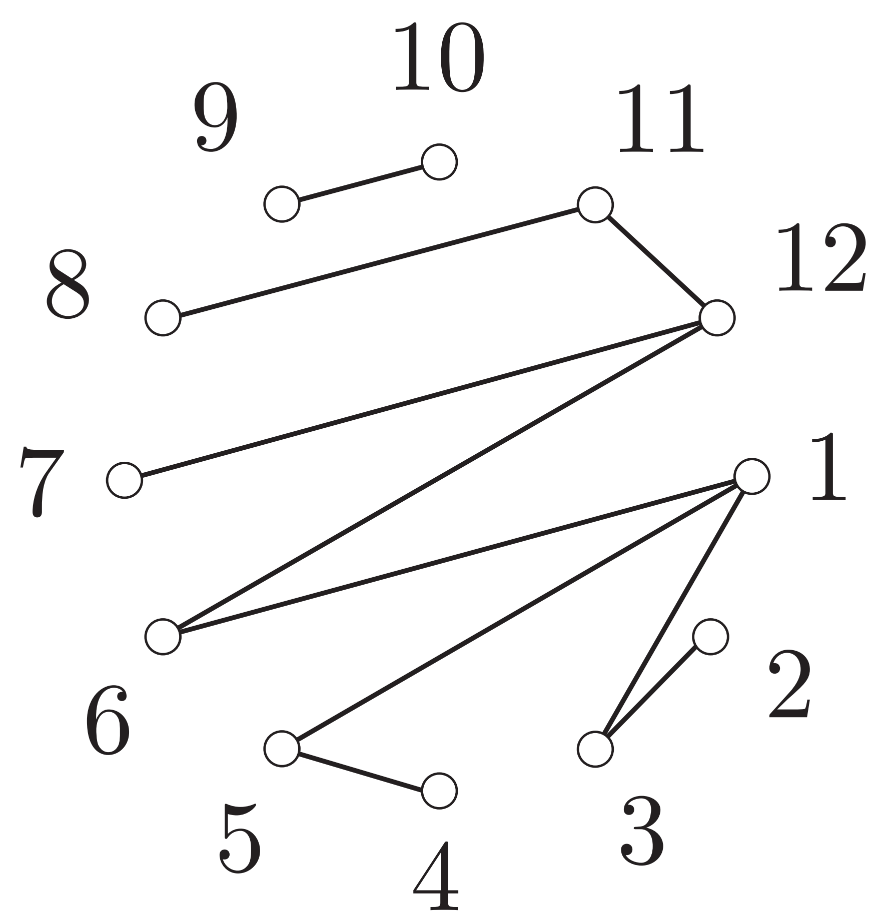
$\mathcal{P}_9$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

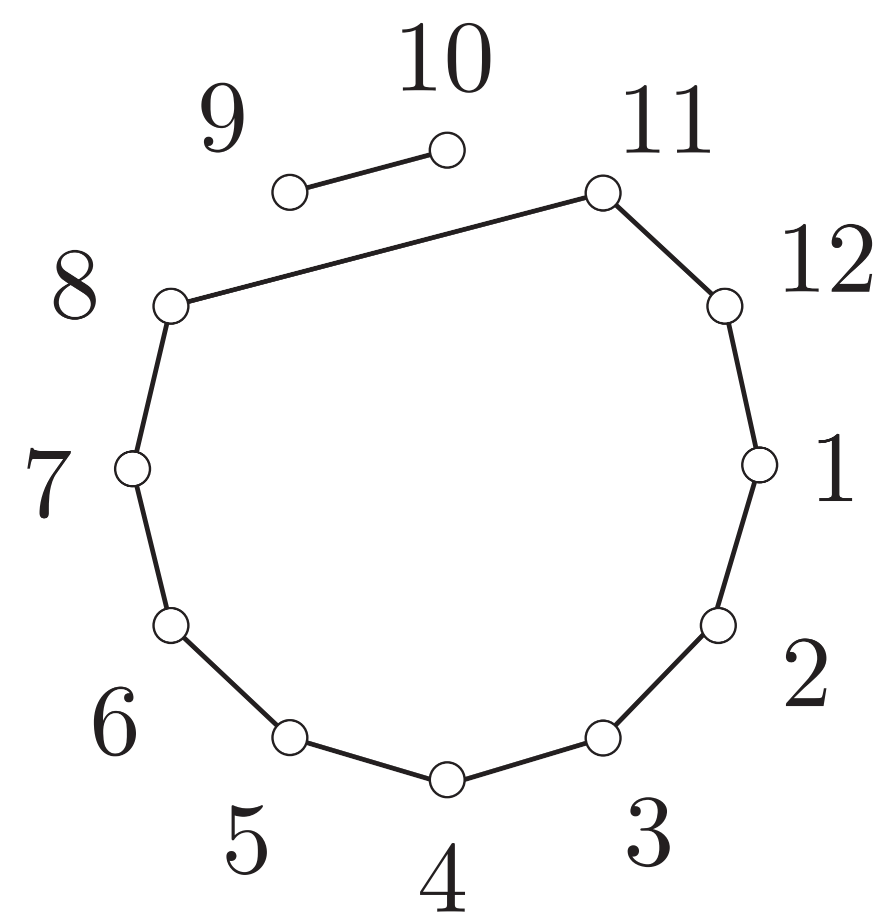
- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1 \tau_2 \cdots \tau_k$.

Example $(n = 12)$. For $k = 10$:

$$\Big(\underbrace{(1,3), (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11)}_{\text{product}=(1,2,3,4,5,6,7,8,11,12)(9,10)}, (9,11) \Big)$$



$\mathcal{F}_{10}$



$\mathcal{P}_{10}$

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- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
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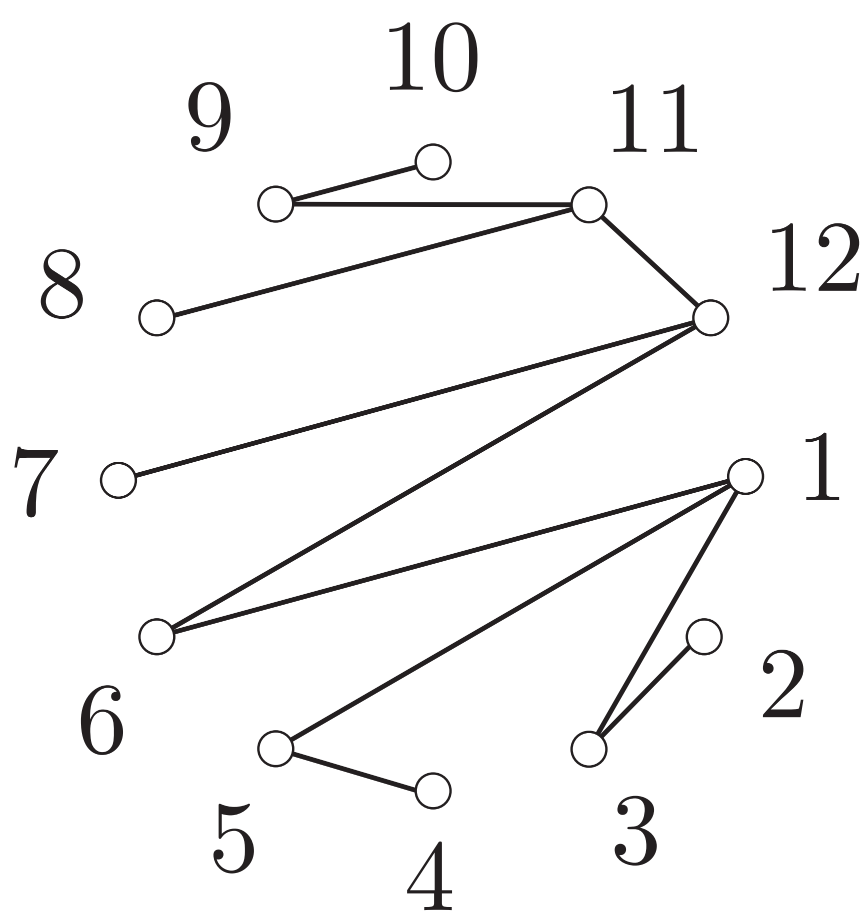
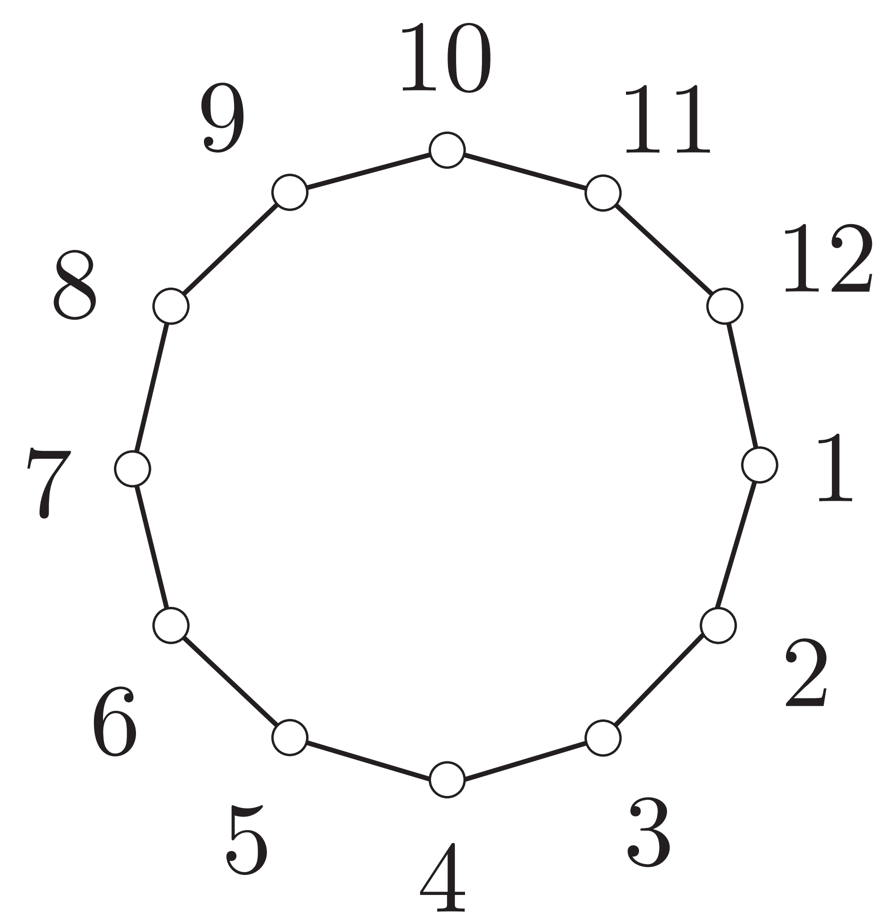
⇝ Example ($n = 12$)). For $k = 11$:

$$\Big(\underbrace{(1,3), (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11), (9,11)}_{\text{product}=(1,2,3,4,5,6,7,8,9,10,11,12)} \Big)$$



$\mathcal{F}_{11}$



$\mathcal{P}_{11}$

Let $(\tau_1^n, \ldots, \tau_{n-1}^n)$ be a uniform minimal factorization of the n-cycle.

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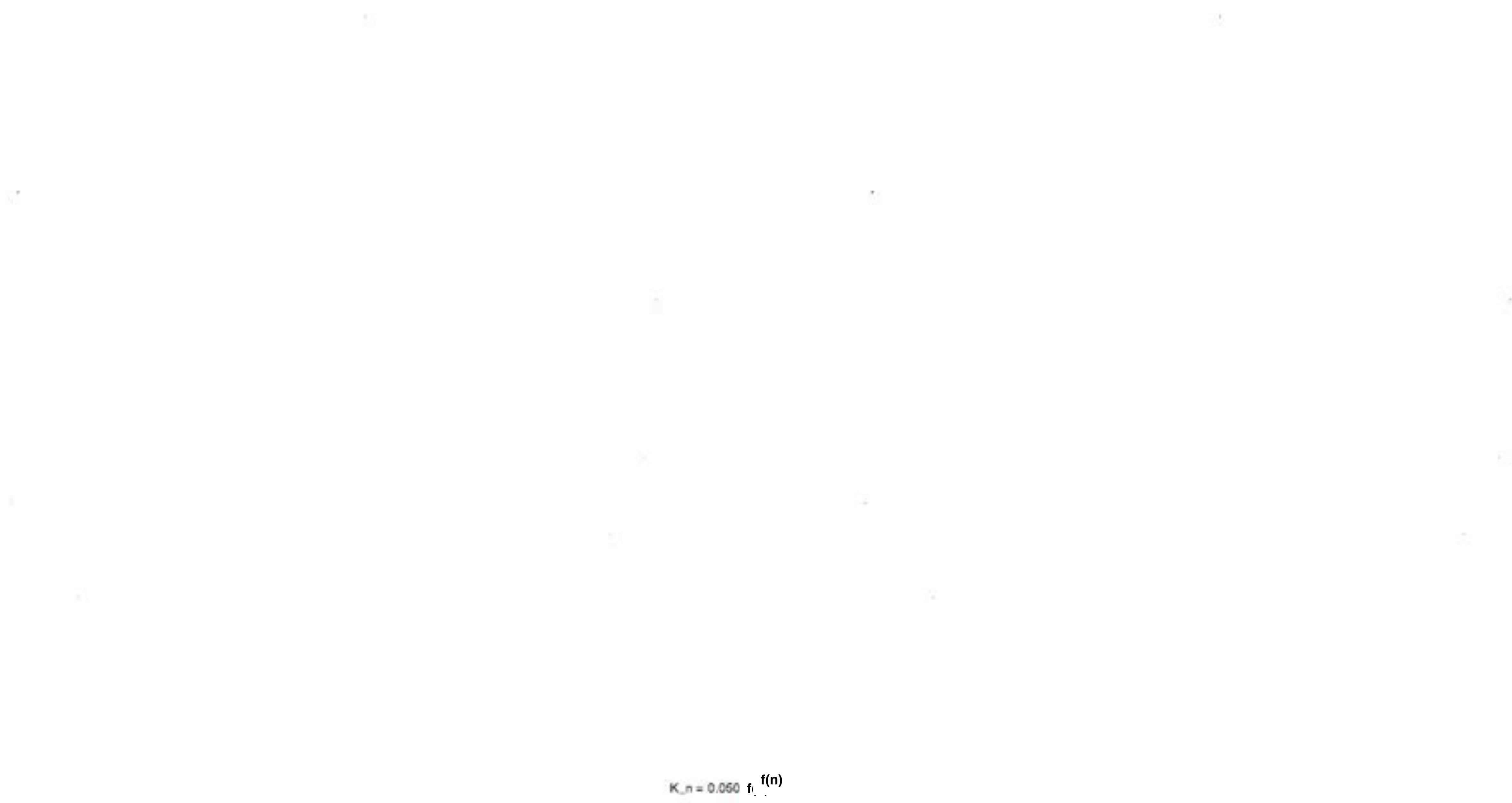
The following film represents

$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right)$$

with $K_n = \lfloor c f(n) \rfloor$ for fixed $n = 20000$, as c varies (for a certain mystery function f).



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Who is f?

The following film represents

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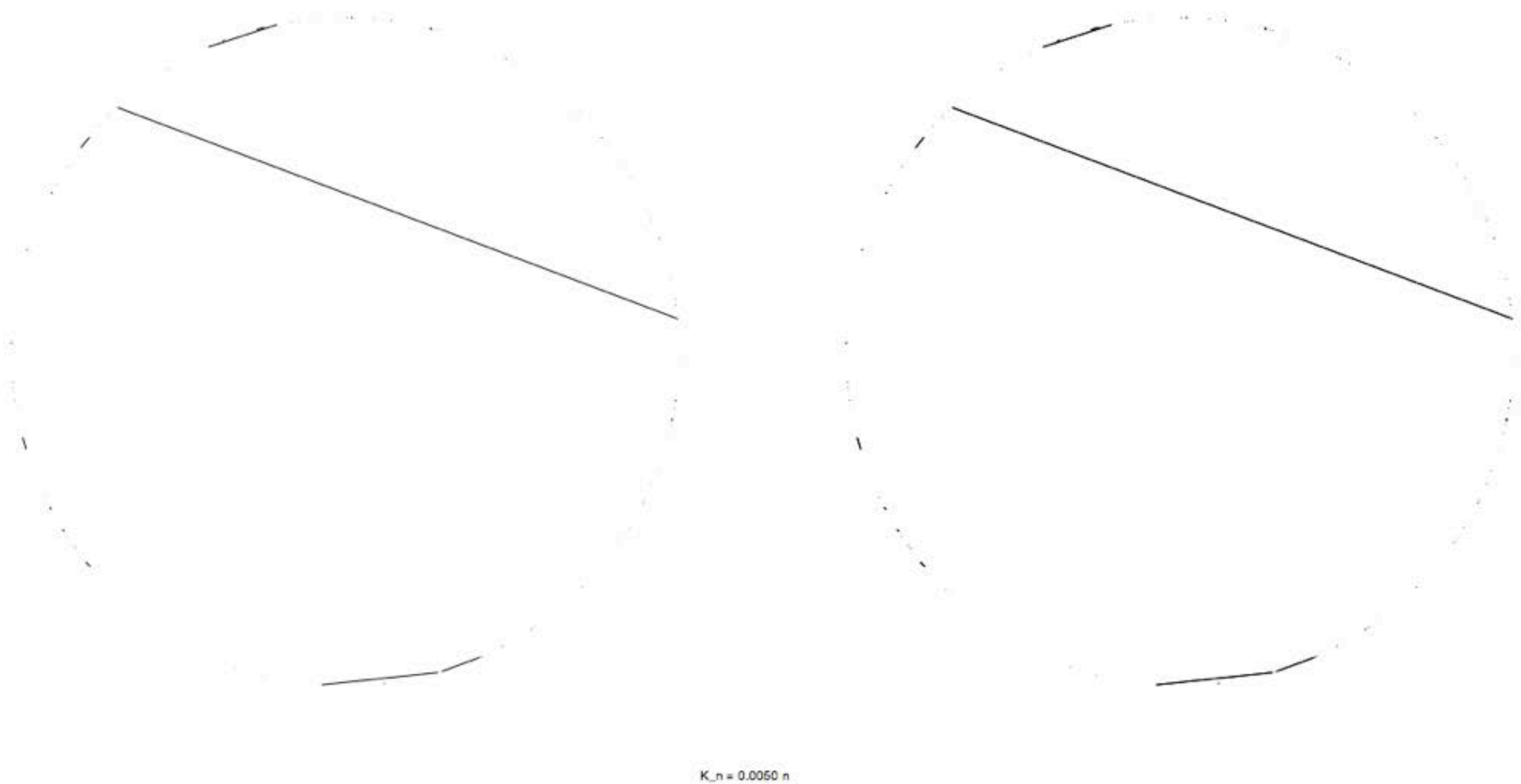
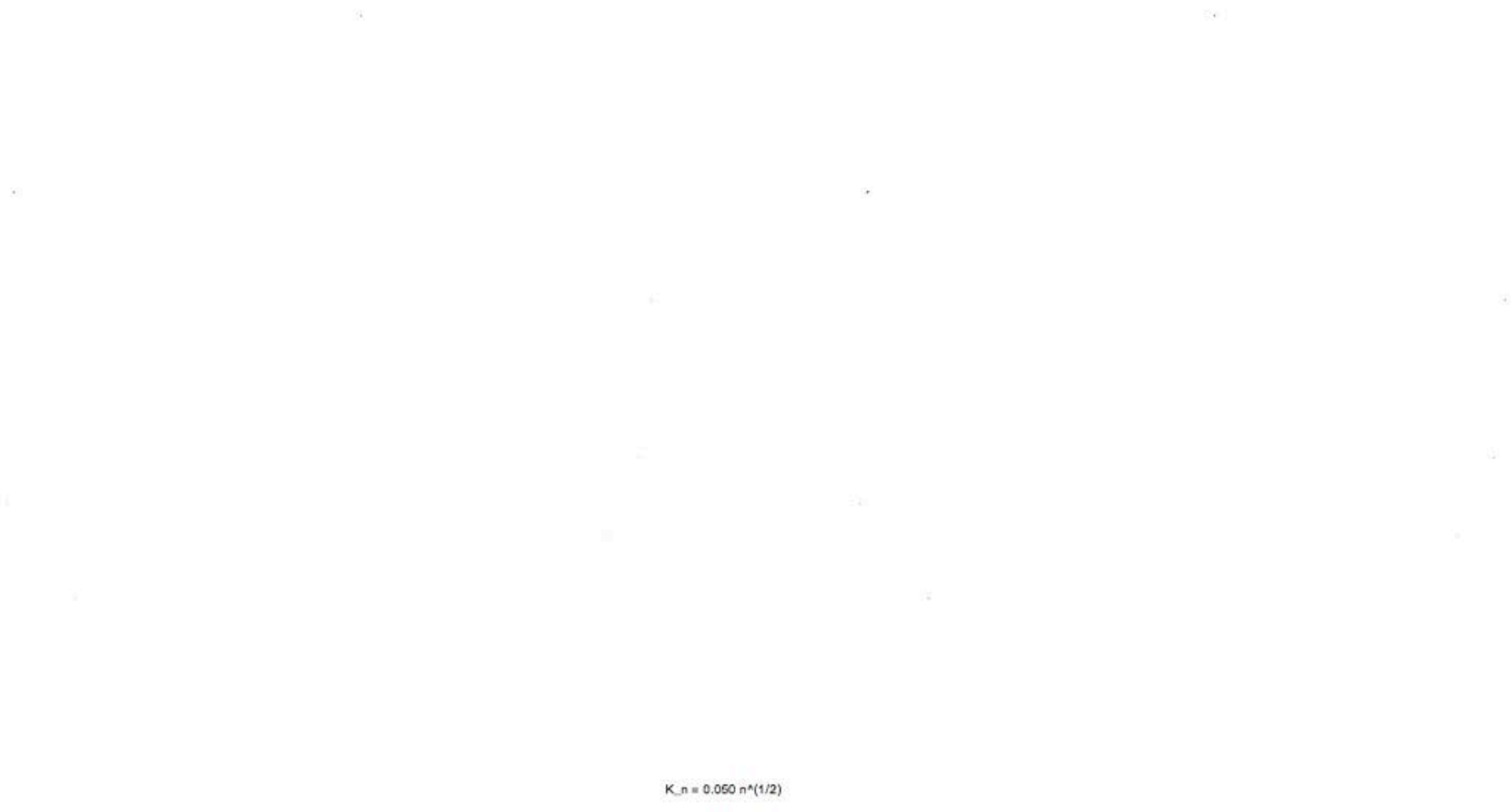
with $\boxed{K_n = \lfloor cn \rfloor}$ for fixed n, as c varies.



The following film represents

$$\left(\mathcal{F}^{n}_{K_n}, \mathcal{P}^{n}_{K_n}\right)$$

with $\boxed{K_n = \lfloor c\sqrt{n} \rfloor}$ for fixed n, as c varies.



Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

(i)

(ii)

(iii)

(iv)

Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

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(i) If $K_n = o(\sqrt{n})$:

(ii) If $\frac{K_n}{\sqrt{n}} \to c \in (0, \infty)$:

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(iv)

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Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

(i) If $K_n = o(\sqrt{n})$: $\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right) \xrightarrow[n\to\infty]{(d)} (\mathbb{S}, \mathbb{S})$.

(ii) If $\frac{K_n}{\sqrt{n}} \to c \in (0, \infty)$:

(iii) If $\frac{K_n}{\sqrt{n}} \to \infty$ and $\frac{n-K_n}{\sqrt{n}} \to \infty$:

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Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

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$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbf{L}_c, \mathbf{L}_c).$$

(iii) If $\frac{K_n}{\sqrt{n}} \to \infty$ and $\frac{n-K_n}{\sqrt{n}} \to \infty$:

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Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

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(iii) If $\frac{K_n}{\sqrt{n}} \to \infty$ and $\frac{n-K_n}{\sqrt{n}} \to \infty$:

$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathrm{L}(\mathbb{e}), \mathrm{L}(\mathbb{e})).$$

(iv) If $\frac{n-K_n}{\sqrt{n}} \to c \in [0, \infty)$:

Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

(i) If $K_n = o(\sqrt{n})$: $\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbb{S}, \mathbb{S})$.

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$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbf{L}_c, \mathbf{L}_c).$$

(iii) If $\frac{K_n}{\sqrt{n}} \to \infty$ and $\frac{n-K_n}{\sqrt{n}} \to \infty$:

$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathrm{L}(\mathbb{e}), \mathrm{L}(\mathbb{e})).$$

(iv) If $\frac{n-K_n}{\sqrt{n}} \to c \in [0, \infty)$:

$$\mathcal{F}^n_{K_n} \quad \xrightarrow[n\to\infty]{(d)} \quad \mathrm{L}(\mathbb{e}), \qquad \mathcal{P}^n_{K_n} \xrightarrow[n\to\infty]{(d)} \quad \mathbf{L}_c \quad (\text{with } \mathbf{L}_0 = \mathbb{S}).$$

ANNALES
HENRI LEBESGUE

What is the limit?

$\rightsquigarrow$ $\mathbf{L}_0$ is the unit circle.

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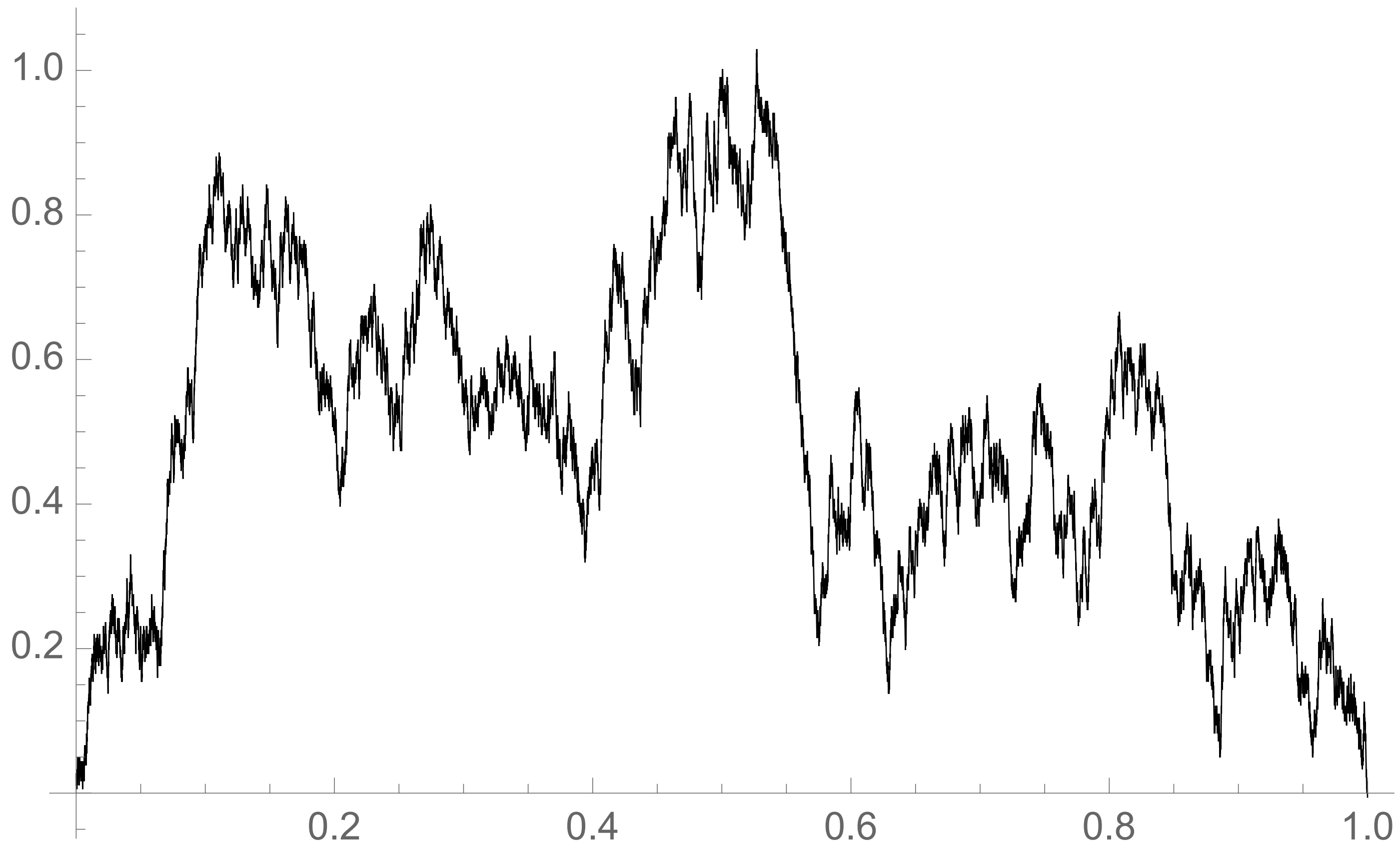

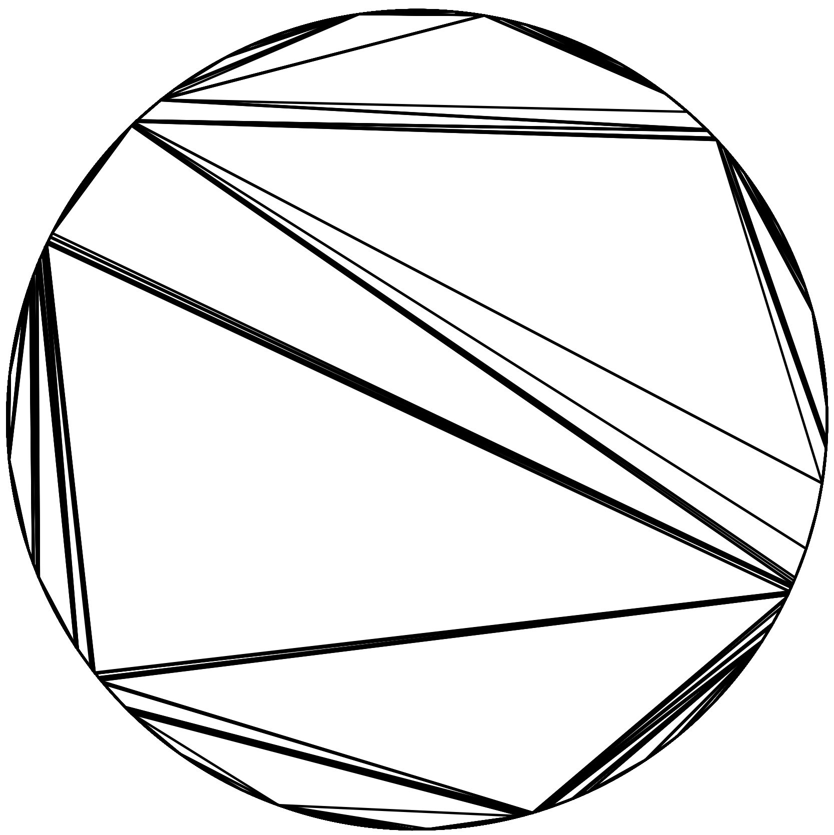

Figure: A Brownian excursion (left) coding $\mathbf{L}_\infty$ (right).

↝ For $0 < \mathrm{c} < \infty$, $\mathbf{L}_{\mathrm{c}}$ is a *lamination*, coded by an excursion of an explicit spectrally positive Lévy process.

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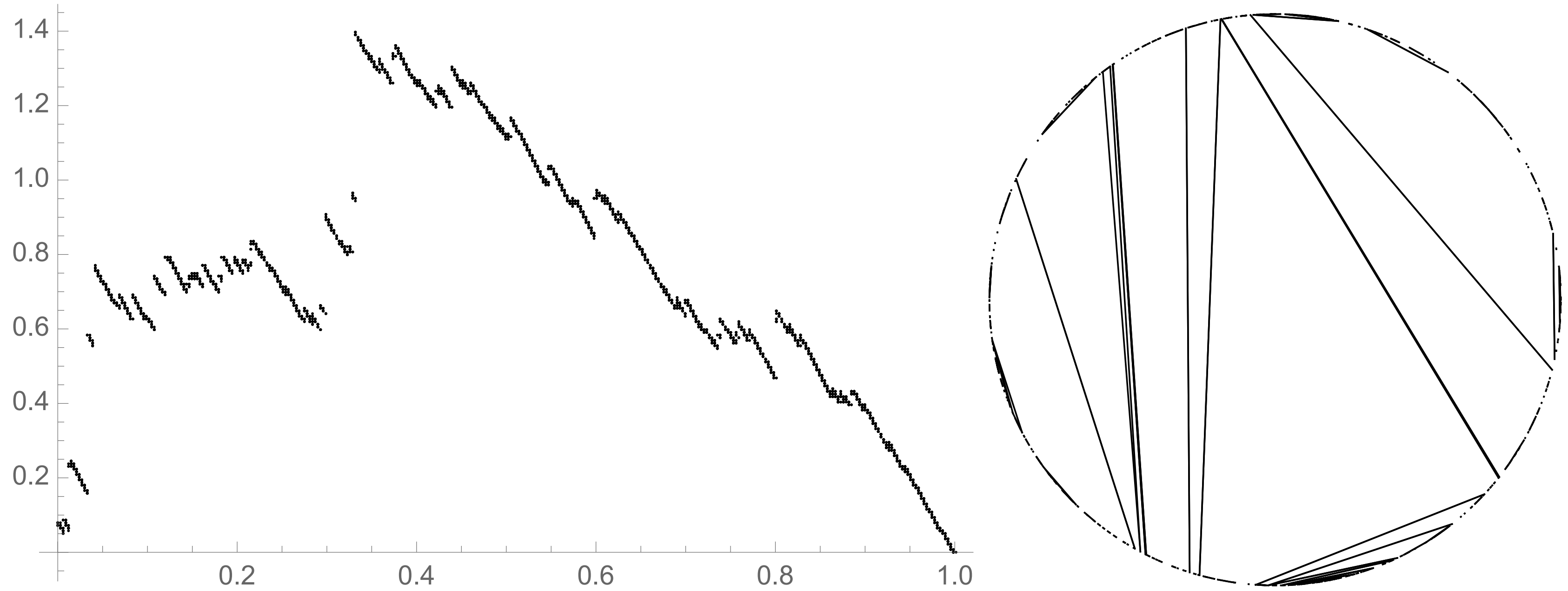

Figure: An excursion of a spectrally positive Lévy process (left) coding $\mathbf{L}_5$ (right).

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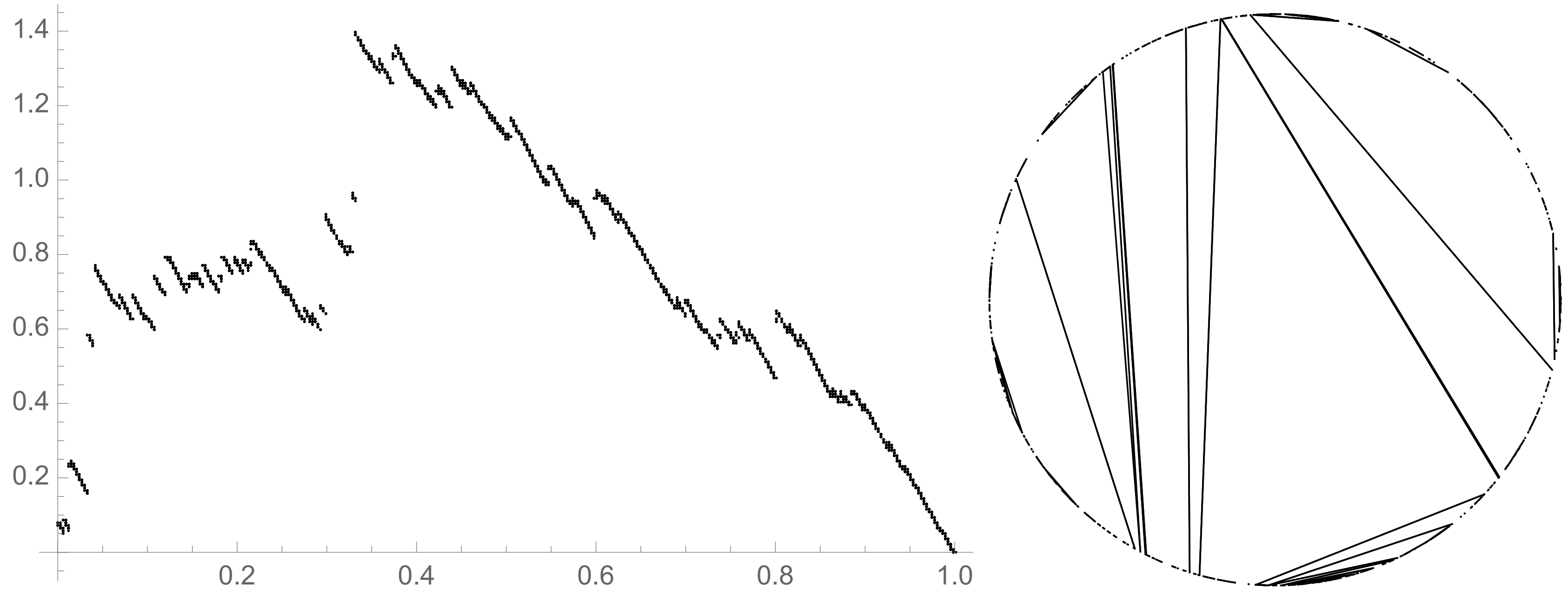

Figure: An excursion of a spectrally positive Lévy process (left) coding $\mathbf{L}_5$ (right).

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$$\Phi(\lambda) = c^2\left(1 - \sqrt{1 + \frac{2\lambda}{c}}\right) + \lambda c.$$

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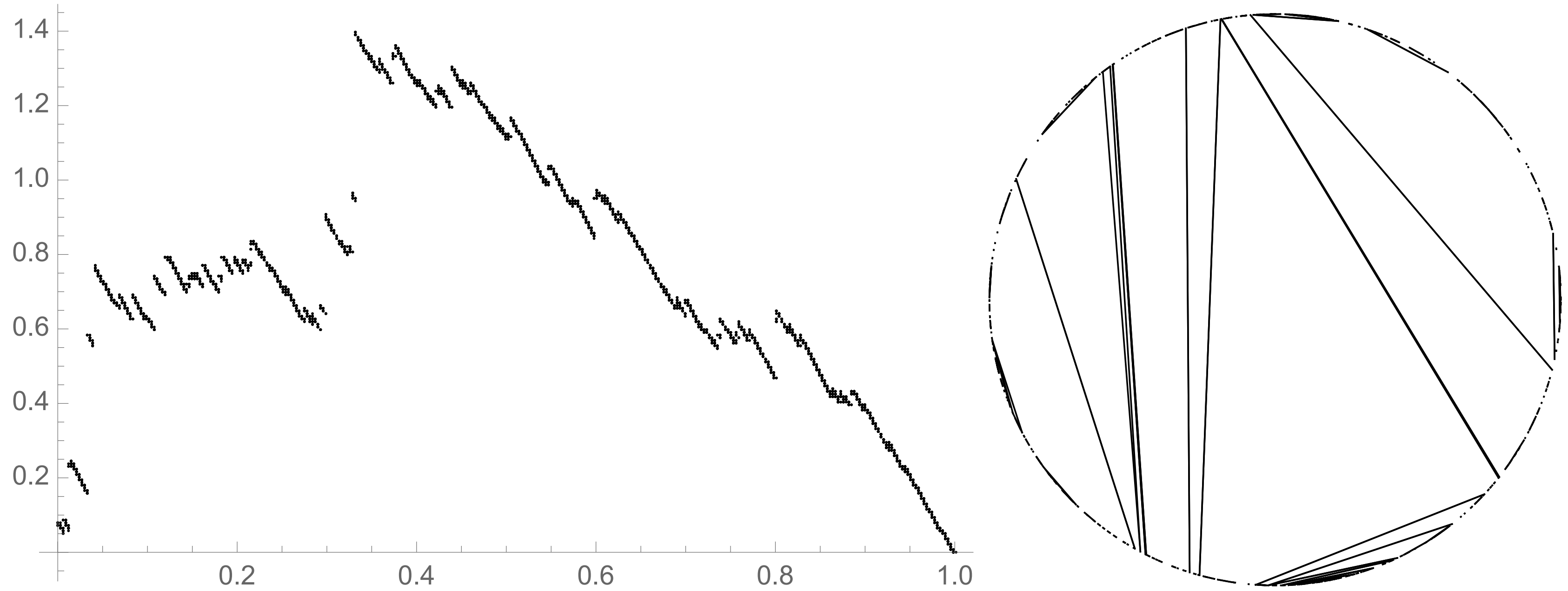

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↝ Thévenin shows the convergence of $\left(\mathcal{F}^n_{\lfloor c\sqrt{n} \rfloor}\right)_{c \geqslant 0}$ to $(\mathbf{L}_c)_{c \geqslant 0}$ as a process.

Main idea of the proof

Proposition (Key fact)**.**

Fix $1 \leqslant k \leqslant n-1$ and let P be a non-crossing partition with n vertices and $n-k$ blocks. Then

$$\mathbb{P}\left(\mathcal{P}(t_1^{(n)} t_2^{(n)} \cdots t_k^{(n)}) = P\right)$$

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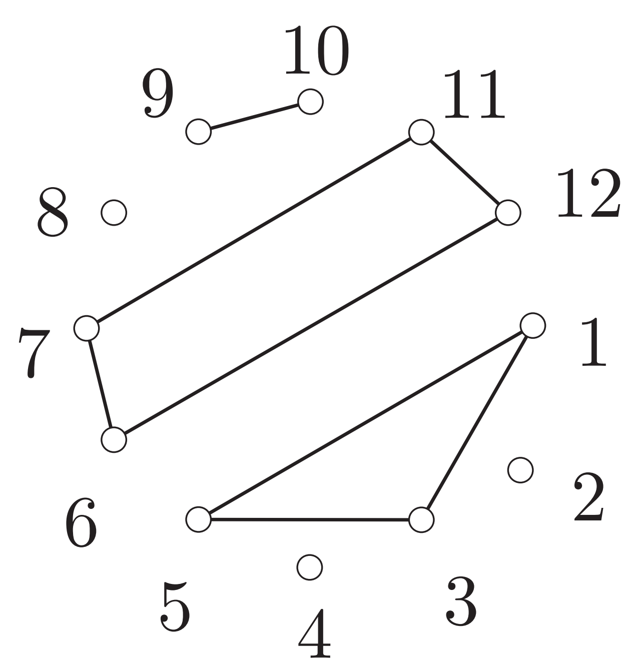

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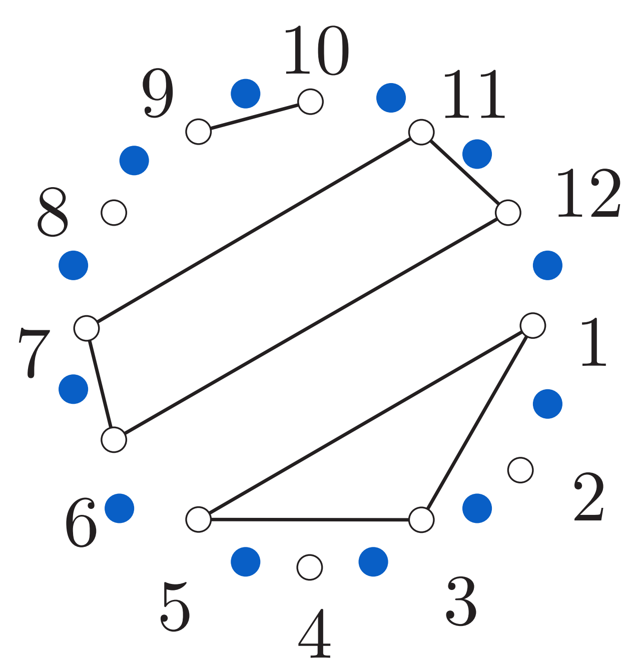

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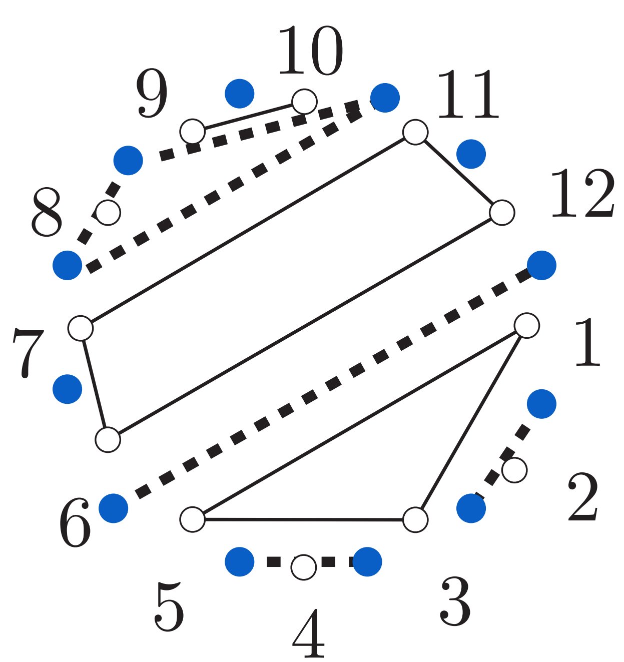

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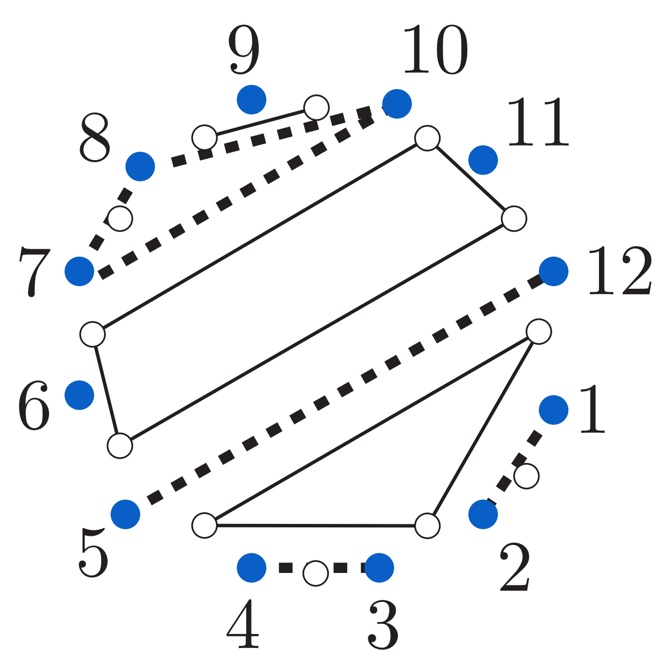

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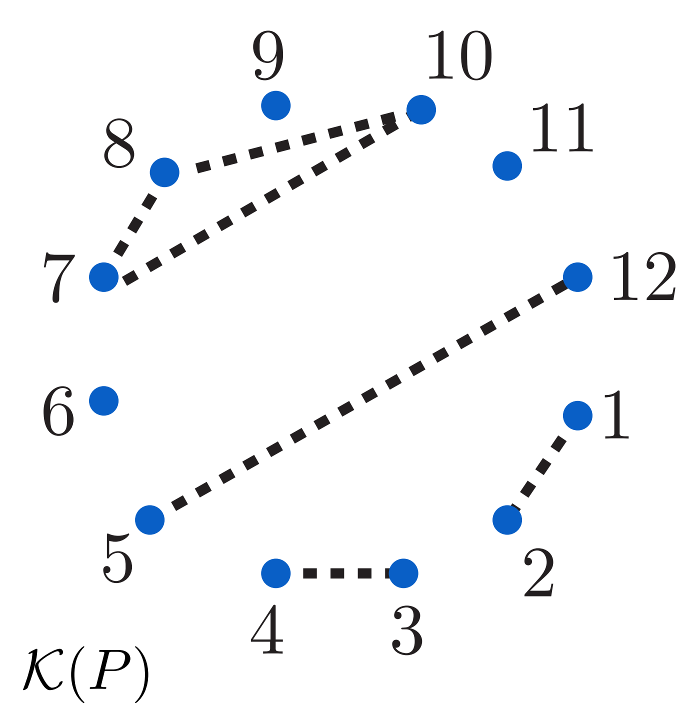

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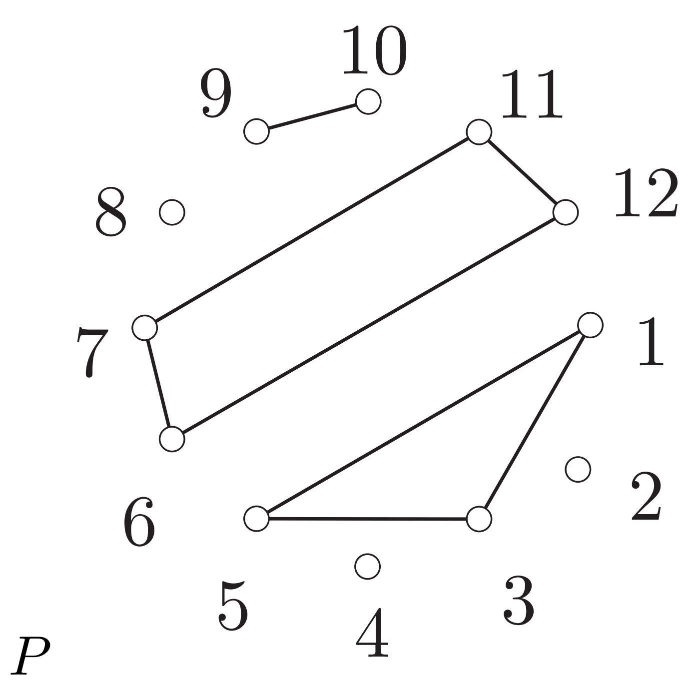

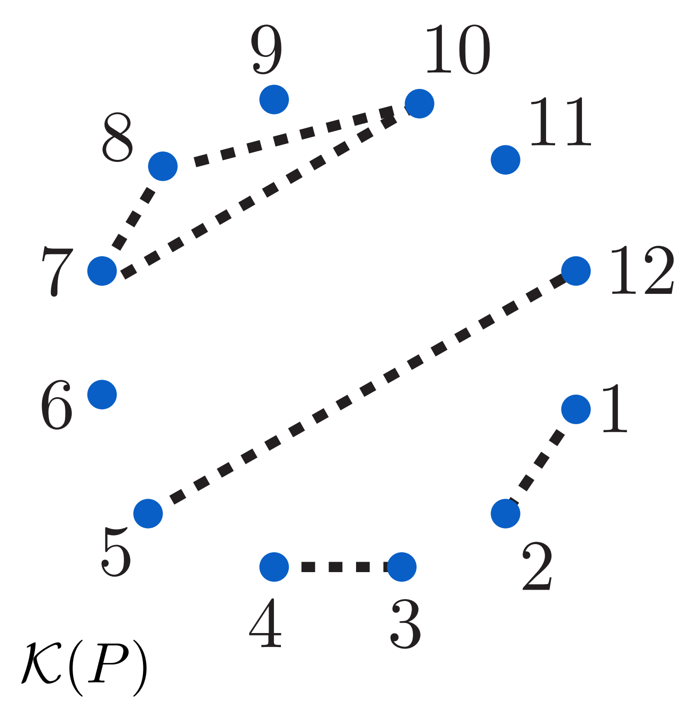

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for n and i large, which explains the $\sqrt{n}$ transition.

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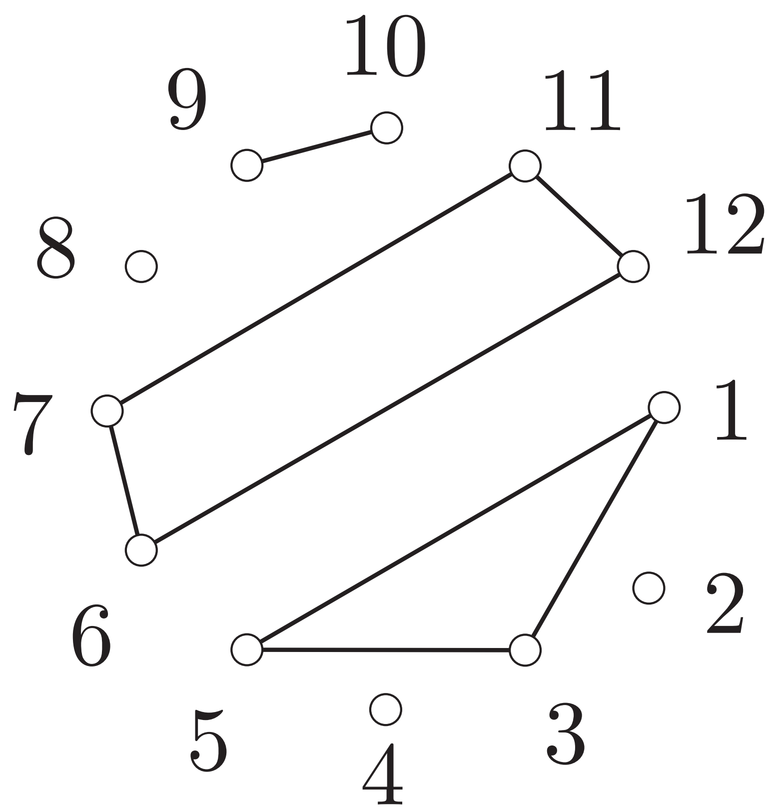

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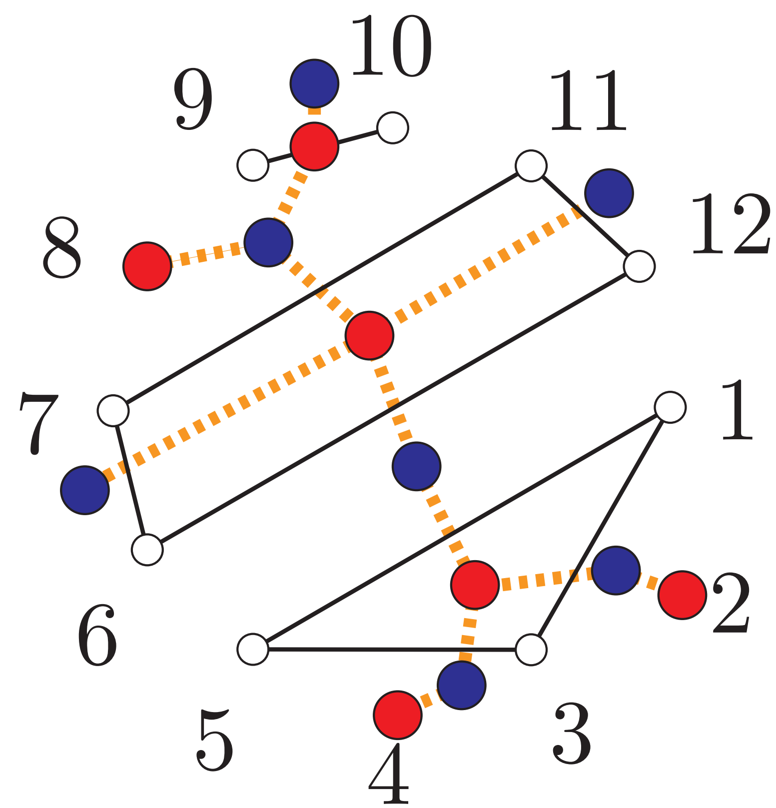

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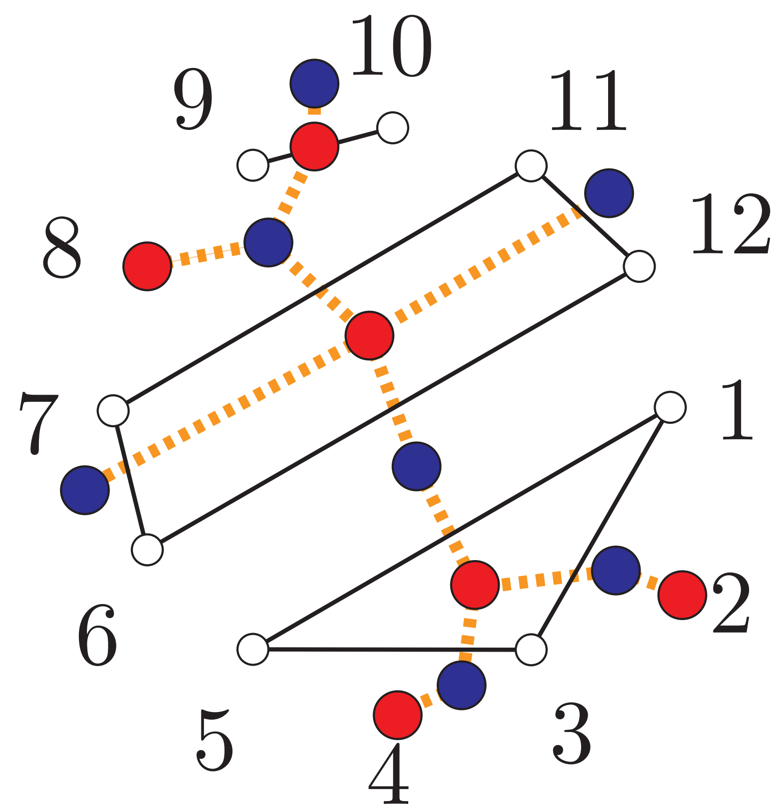

It follows that $\mathcal{P}(t_1^{(n)} t_2^{(n)} \cdots t_k^{(n)})$ is coded by a bitype biconditioned Bienaymé–Galton–Watson (or simply generated) tree ($n-k$ blue vertices and $k+1$ red vertices)!

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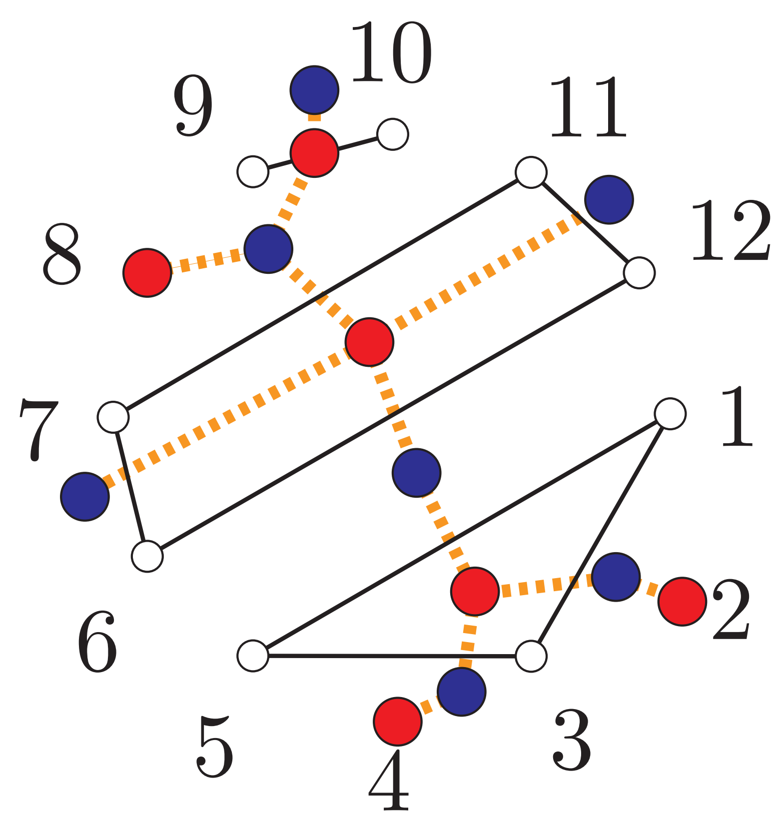

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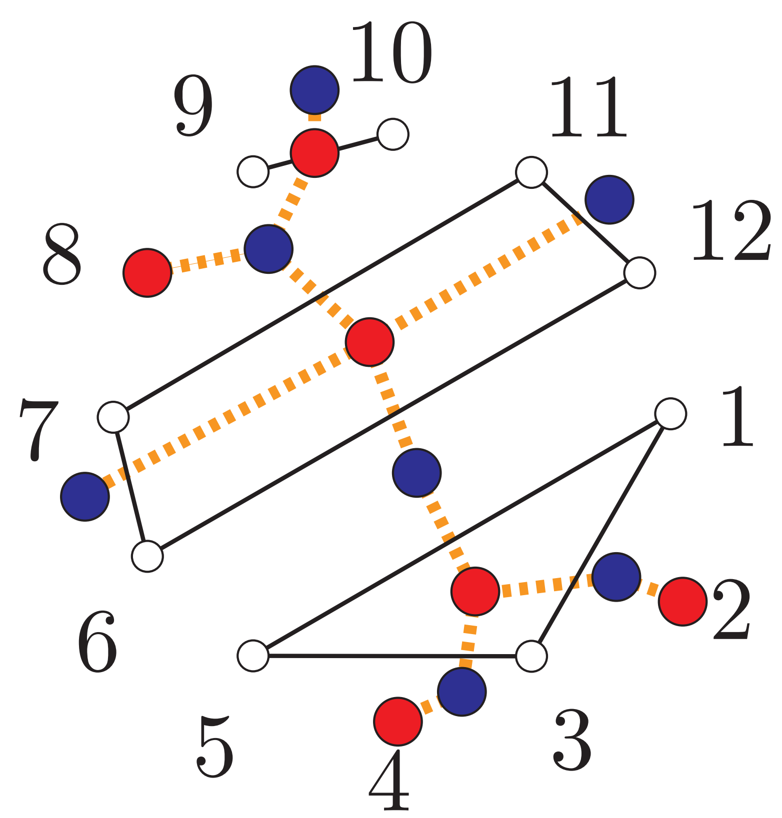

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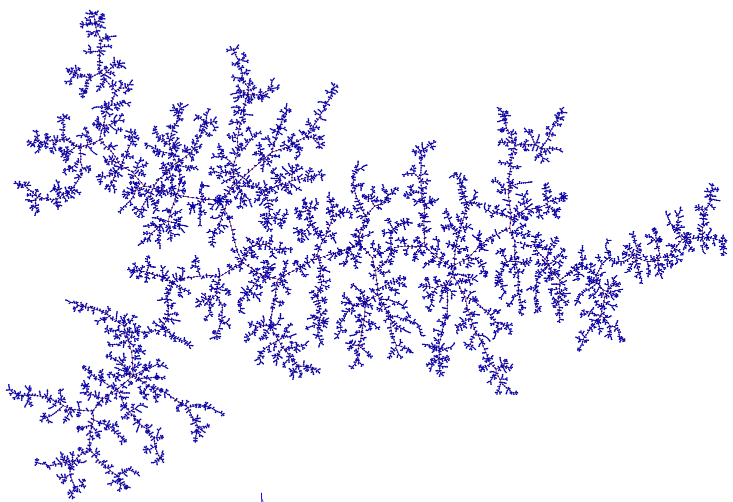
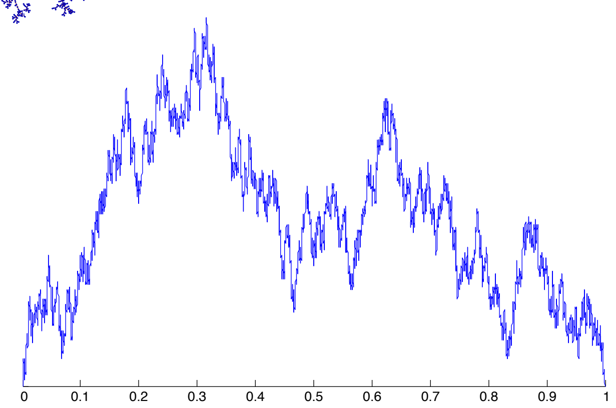
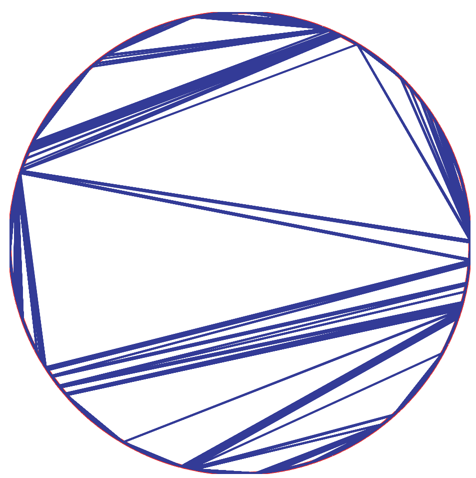
We develop a new machinery to study limits of such random trees.







K_n = 0.050 n^(1/2)