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Proof of the local estimate

Outline: 1) A maximal inequality
2) The local estimate

1) A maximal inequality

Theorem [Fuk-Nagaev '71, Denisov-Dieker-Schneer '08]
Assume that X is a $\mathbb{R}$-valued rv with $\mathbb{E}[X^2]<\infty$, $\mathbb{E}[X]=0$. Let $(X_i)_{i\geq 1}$ be iid having same law as X. Set $S_n = X_1+\cdots+X_n$. There $\exists K>0$ s.t $\forall n\geq 1$, $x>0$ and $c\geq 1$:

$$\mathbb{P}(S_n > x\sqrt{n}, X_1\leq c\sqrt{n}, \ldots, X_n \leq c\sqrt{n}) \leq K\exp\left(-\frac{x}{c}\right)$$

Proof: The idea is to introduce the truncated random walk $\tilde{S}_n = \sum_{i=1}^{n} X_i \mathbb{1}_{X_i\leq c\sqrt{n}}$. Indeed,
$\mathbb{P}(S_n > x\sqrt{n}, X_1\leq c\sqrt{n}, \ldots, X_n \leq c\sqrt{n}) \leq \mathbb{P}(\tilde{S}_n \geq x\sqrt{n})$.

To bound this probability we use the "exponential Markov" inequality:

$$\mathbb{P}(\tilde{S}_n \geq x\sqrt{n}) = \mathbb{P}\left(e^{\frac{\tilde{S}_n}{c\sqrt{n}}} \geq e^{\frac{x}{c}}\right) \leq e^{-\frac{x}{c}}\, \mathbb{E}\left[e^{\frac{\tilde{X}}{c\sqrt{n}}}\right]^n \quad \text{with } \tilde{X} = X\mathbb{1}_{X\leq c\sqrt{n}}$$

(⚠ $\tilde{X}$ depends on n)

We show that $\mathbb{E}\left[e^{\frac{\tilde{X}}{c\sqrt{n}}}\right] = 1 + O\left(\frac{1}{n}\right)$ and the result will follow.

The idea is to write $e^x = 1+x+x^2 r(x)$ with $r(x)=\frac{e^x-1-x}{x^2}$, so that

$$\mathbb{E}\left[e^{\tilde{X}/c\sqrt{n}}\right] = 1 + \underbrace{\mathbb{E}\left[\frac{\tilde{X}}{c\sqrt{n}}\right]}_{m_n} + \underbrace{\mathbb{E}\left[\left(\frac{\tilde{X}}{c\sqrt{n}}\right)^2 r\left(\frac{\tilde{X}}{c\sqrt{n}}\right)\right]}_{s_n}$$

We show that $m_n = O\left(\frac{1}{n}\right)$ and $s_n = O\left(\frac{1}{n}\right)$

- For s_n: since r is bounded on $(-\infty,1]$, we have $s_n = O\left(\mathbb{E}\left[\frac{\tilde{X}^2}{n}\right]\right)$ (here we use $c\geq 1$)
But $\mathbb{E}[\tilde{X}^2]\leq \mathbb{E}[X^2]$, thus $s_n = O\left(\frac{1}{n}\right)$.
- For m_n: write $m_n = \frac{1}{c\sqrt{n}}\mathbb{E}[X\mathbb{1}_{X\leq c\sqrt{n}}] = -\frac{1}{c\sqrt{n}}\mathbb{E}[X\mathbb{1}_{X>c\sqrt{n}}]$ because $\mathbb{E}[X]=0$
But $\mathbb{E}[|X|\mathbb{1}_{|X|>c\sqrt{n}}] \leq \mathbb{E}\left[|X|\cdot\frac{|X|}{c\sqrt{n}}\mathbb{1}_{|X|>c\sqrt{n}}\right] \leq \frac{1}{c\sqrt{n}}\mathbb{E}[X^2]$
Thus $m_n = O\left(\frac{1}{n}\right)$ (here we use $c\geq 1$)

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Remark The result is false with "$c>0$" instead of "$c\geq 1$". Indeed, take $\mathbb{P}(X_1=\pm 1)=\frac{1}{2}$, $x=1$, $c=n^{1/4}$.
Then $\mathbb{P}(S_n\geq x\sqrt{n}, X_1\leq c\sqrt{n},\dots,X_n\leq c\sqrt{n}) = \mathbb{P}(S_n\geq\sqrt{n}) \underset{n\to\infty}{\longrightarrow} \mathbb{P}(\mathcal{N}(0,1)\geq 1)$
but $K\exp(-\frac{x}{c}) \underset{n\to\infty}{\longrightarrow} 0$

2) The local estimate

Let $(X_i)_{i\geq 1}$ be iid real-valued random variables. Set $S_0=0$ and $S_n=X_1+\dots+X_n$ for $n\geq 1$.

Assumption (H) $\mathbb{E}[X_1^2]<\infty$ and there exist $c>0$ and $\beta>2$ such that $\mathbb{P}(X_1\in[u,u+1)) \underset{u\to\infty}{\sim} \frac{c}{u^{1+\beta}}$

It is not difficult to check that under (H), $\mathbb{P}(X_1\geq u) \underset{u\to\infty}{\sim} \frac{c/\beta}{u^\beta}$ and that $\mathbb{E}[X_1^2]<\infty$.

Theorem (Doney '89, Nagaev '57)
Assume that X_1 satisfies (H) and that $\mathbb{E}[X_1]=0$. Fix $\varepsilon>0$. Then, uniformly in $m\geq\varepsilon n$,
$$\mathbb{P}(S_n\in[m,m+1)) \underset{n\to\infty}{\sim} n\,\mathbb{P}(X_1\in[m,m+1))$$
That is, $\sup_{m\geq\varepsilon n}\left|\frac{\mathbb{P}(S_n\in[m,m+1))}{n\,\mathbb{P}(X_1\in[m,m+1))}-1\right| \underset{n\to\infty}{\longrightarrow} 0$

We will use several times the following fact

Fact (*) Under (H), $\mathbb{P}(X_1\in[u,u+1)) \underset{u\to\infty}{\sim} \mathbb{P}(X_1\in[u+y,u+y+1))$ uniformly in $|y|\leq\frac{u}{\ln(u)}$,
i.e. $\sup_{|y|\leq\frac{u}{\ln(u)}}\left|\frac{\mathbb{P}(X_1\in[u,u+1))}{\mathbb{P}(X_1\in[u+y,u+y+1))}-1\right| \underset{u\to\infty}{\longrightarrow} 0$

Proof Set $\bar{m}=\frac{m}{\ln(m)^3}$. Write $n P_1^{m,n}\leq\mathbb{P}(S_n\in[m,m+1))\leq nP_1^{m,n}+P_0^{m,n}+P_2^{m,n}$ with

- $P_1^{m,n}=\mathbb{P}(S_n\in[m,m+1), X_n\geq\bar{m}, \max_{1\leq k\leq n-1}X_k<\bar{m})$
- $P_0^{m,n}=\mathbb{P}(S_n\in[m,m+1), \max_{1\leq k\leq n}X_k<\bar{m})$
- $P_2^{m,n}=\mathbb{P}(S_n\in[m,m+1), \bigcup_{1\leq j<k\leq n}\{X_j\geq\bar{m}, X_k\geq\bar{m}\})$

We show that uniformly in $m\geq\varepsilon n$, $P_1^{m,n}\sim\mathbb{P}(X_1\in[m,m+1))$, $P_0^{m,n}=o(n\mathbb{P}(X_1\in[m,m+1)))$, $P_2^{m,n}=o(n\mathbb{P}(X_1\in[m,m+1)))$

Recall that as $m\to\infty$, $\mathbb{P}(X_1\in[m,m+1))\sim\frac{c}{m^{1+\beta}}$ and $\mathbb{P}(X_1\geq m)\sim\frac{c_0}{m^\beta}$ with $c_0=c/\beta$ Lemma.

For $P_2^{m,n}$: $P_2^{m,n}\leq\binom{n}{2}\mathbb{P}(X_1\geq\bar{m}, X_2\geq\bar{m})$
$$\overset{(H_0)}{\leq} C'n^2\frac{1}{\bar{m}^\beta}\cdot\frac{1}{\bar{m}^\beta}=\frac{n^2}{m^{2\beta}}\cdot\ln(m)^{6\beta}, \quad\text{so}\quad \frac{P_2^{m,n}}{n\,\mathbb{P}(X_1\in[m,m+1))}\leq C'\frac{n}{m^{2\beta}}\cdot\ln(m)^{6\beta}\cdot m^{1+\beta}=C'\frac{n\cdot\ln(m)^{6\beta}}{m^{\beta-1}}\longrightarrow 0$$
because $\beta-1>1$

For $P_0^{m,n}$: We use the maximal inequality:

$$P_0^{m,n} \leq \mathbb{P}(S_n \geq m, \max_{1\leq k\leq n} X_k < \overline{m}) \leq K \exp(-\frac{m}{\overline{m}}) = K \exp(-\ln(m)^3) = o\left(\frac{n}{m^{1+\beta}}\right)$$

For $P_1^{m,n}$: This is more delicate, we need to consider cases according to the value of S_{n-1}.

Since $\frac{S_n}{\sqrt{n}}$ converges in distribution (CLT), $\frac{S_n}{n^{3/4}} \xrightarrow{\mathbb{P}} 0$. Then write

$$P_1^{m,n} = Q_1^{m,n} + Q_2^{m,n} + Q_3^{m,n} \text{ with}$$

$$Q_1^{m,n} = \mathbb{P}(S_n \in [m,m+1), X_n \geq \overline{m}, \max_{1\leq k\leq n-1} X_k < \overline{m}, S_{n-1} > \frac{m}{\ln(m)})$$

$$Q_2^{m,n} = \mathbb{P}(S_n \in [m,m+1), X_n \geq \overline{m}, \max_{1\leq k\leq n-1} X_k < \overline{m}, -n^{3/4} < S_{n-1} < \frac{m}{\ln(m)})$$

$$Q_3^{m,n} = \mathbb{P}(S_n \in [m,m+1), X_n \geq \overline{m}, \max_{1\leq k\leq n-1} X_k < \overline{m}, S_{n-1} < -n^{3/4})$$

We first show $Q_1^{m,n}, Q_3^{m,n} = o(\mathbb{P}(X_1 \in [m,m+1)))$:

- For $Q_1^{m,n}$: By the maximal inequality, $Q_1^{m,n} \leq \mathbb{P}(S_{n-1} > \frac{m}{\ln(m)}, \max_{1\leq k\leq n-1} X_k < \overline{m}) \leq K \exp(-\ln(m)^2) = o(\mathbb{P}(X_1 \in \Delta_m))$.
- For $Q_3^{m,n}$:

$$Q_3^{m,n} \leq \sum_{i<-n^{3/4}} \mathbb{P}(S_{n-1} = i, X_n \in \Delta_{m-i})$$
$$= \sum_{i<-n^{3/4}} \mathbb{P}(S_{n-1} = i)\mathbb{P}(X_n \in \Delta_{m-i})$$
$$\leq \underbrace{\sum_{i<-n^{3/4}} \mathbb{P}(S_{n-1} = i)}_{=\mathbb{P}(S_{n-1} < -n^{3/4}) = o(1)} \underbrace{\sup_{j\geq n^{3/4}} \mathbb{P}(X_1 \in [m+j, m+j+1))}_{= O(\mathbb{P}(X_1 \in [m,m+1))) \text{ by } (H_\Delta)}$$

- For $Q_2^{m,n}$:

$$Q_2^{m,n} = \mathbb{P}(\max_{1\leq k\leq n-1} X_k < \overline{m}, -n^{3/4} < S_{n-1} < \frac{m}{\ln(m)}, X_n \geq \overline{m}, X_n \in [m-S_{n-1}, m-S_{n-1}+1))$$

$$= \mathbb{P}(\max_{1\leq k\leq n-1} X_k < \overline{m}, -n^{3/4} < S_{n-1} < \frac{m}{\ln(m)}, X_n \in [m-S_{n-1}, m-S_{n-1}+1))$$

because $m - \frac{m}{\ln(m)} \geq \overline{m}$ for n large enough.

Observe that by assumption $\mathbb{P}(X_n \in [m-j, m-j+1)) \sim \mathbb{P}(X_n \in [m,m+1))$ uniformly for $-n^{3/4} < j < \frac{m}{\ln(m)}$ (by fact (*))

Thus $Q_2^{m,n} \underset{n\to}{\sim} \mathbb{P}(\max_{1\leq k\leq n-1} X_k < \overline{m}, -n^{3/4} < S_{n-1} < \frac{m}{\ln(m)}) \mathbb{P}(X_1 \in [m,m+1))$

It remains to check that:

- $\mathbb{P}(-n^{3/4} < S_{n-1} < \frac{m}{\ln(m)}) \xrightarrow[n\to\infty]{} 1$, which is clear since $\frac{S_{n-1}}{\sqrt{n}}$ converges in distribution
- $\mathbb{P}(\max_{1\leq k\leq n-1} X_k < \overline{m}) \xrightarrow[n\to\infty]{} 1$;

(indeed, for two event A, B, $\mathbb{P}(A\cap B) = 1 - \mathbb{P}(A\cup B) \geq 1 - \mathbb{P}(A) - \mathbb{P}(B) = 1 - \mathbb{P}(A^c) - \mathbb{P}(B^c)$)

this comes from the fact that $\mathbb{P}(\max_{1\leq k\leq n-1} X_k < \overline{m}) = (1 - \mathbb{P}(X_1 > \overline{m}))^{n-1} = \exp(-n\, \mathbb{P}(X_1 > \overline{m})(1+o(1)))$

and $n\,\mathbb{P}(X_1 > \overline{m}) \sim c' \frac{n}{m^\beta} \cdot \ln(m)^{3\beta} \xrightarrow[n\to\infty]{} 0$ since $\beta > 2$

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Remark There has been quite some work to find the "best" sequence m_n such that the theorem holds uniformly for $m \geq m_n$ (we have shown that $m_n = \varepsilon n$ works).