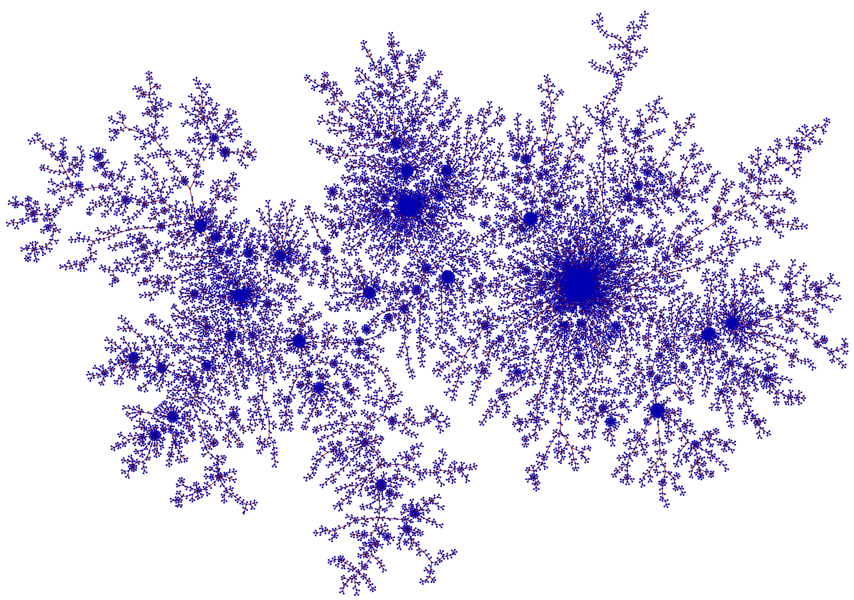
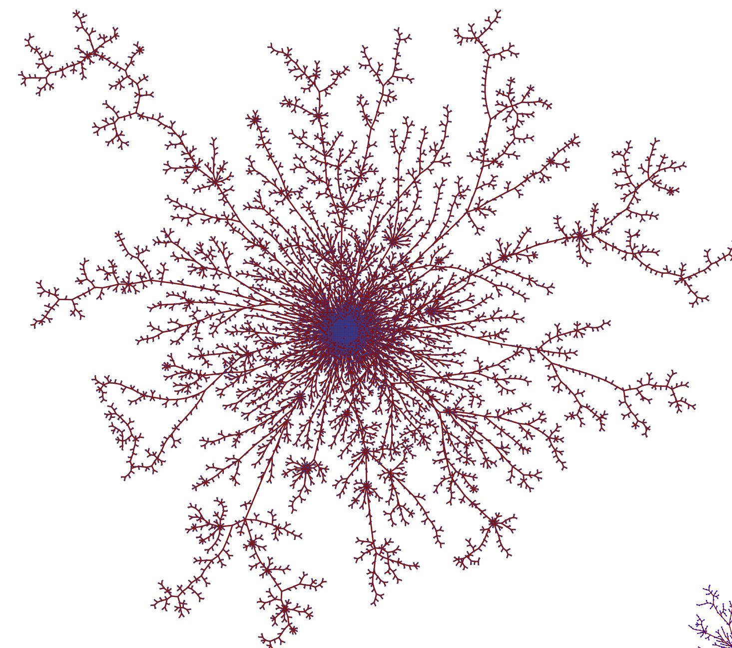
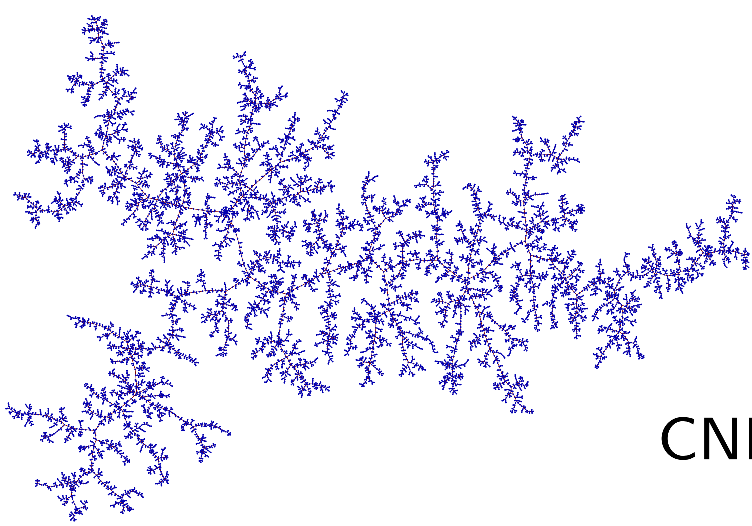
Limits of large random trees







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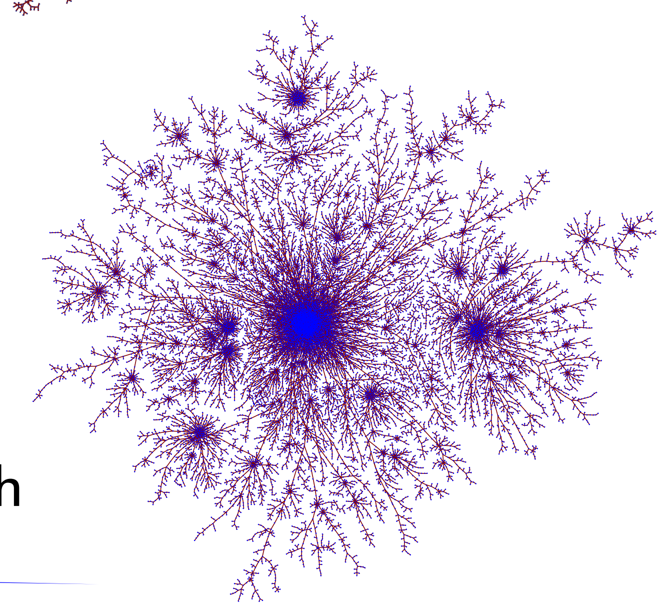

Condensation Phenomena in Random Trees – 2024 CRM-PIMS Summer School in Probability

Context

Understand the geometry and the structure of large random trees by studying their limits.

Motivation for studying limits

Let $(X_n)_{n \geqslant 1}$ be "discrete" objects converging towards a "limiting" object X:

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by X and passes to the limit, X_n satisfies "approximately" $\mathcal{P}$ for n large.

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by X and passes to the limit, X_n satisfies "approximately" $\mathcal{P}$ for n large.
- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards X, then X_n and Y_n share approximately the same properties for n large.

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$$X_n \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} X \quad \text{implies} \quad G(X_n) \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} G(X)$$

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Outline

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Stack triangulations (Albenque, Marckert)

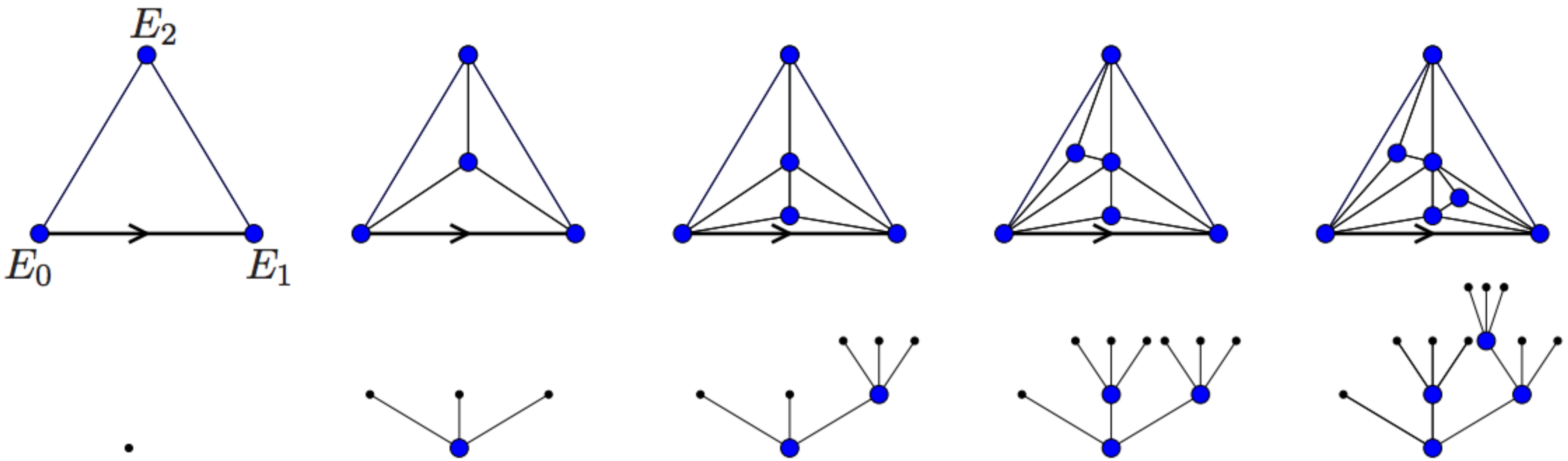


Figure 8: Construction of the ternary tree associated with an history of a stack-triangulation

Dissections (Curien, K.)

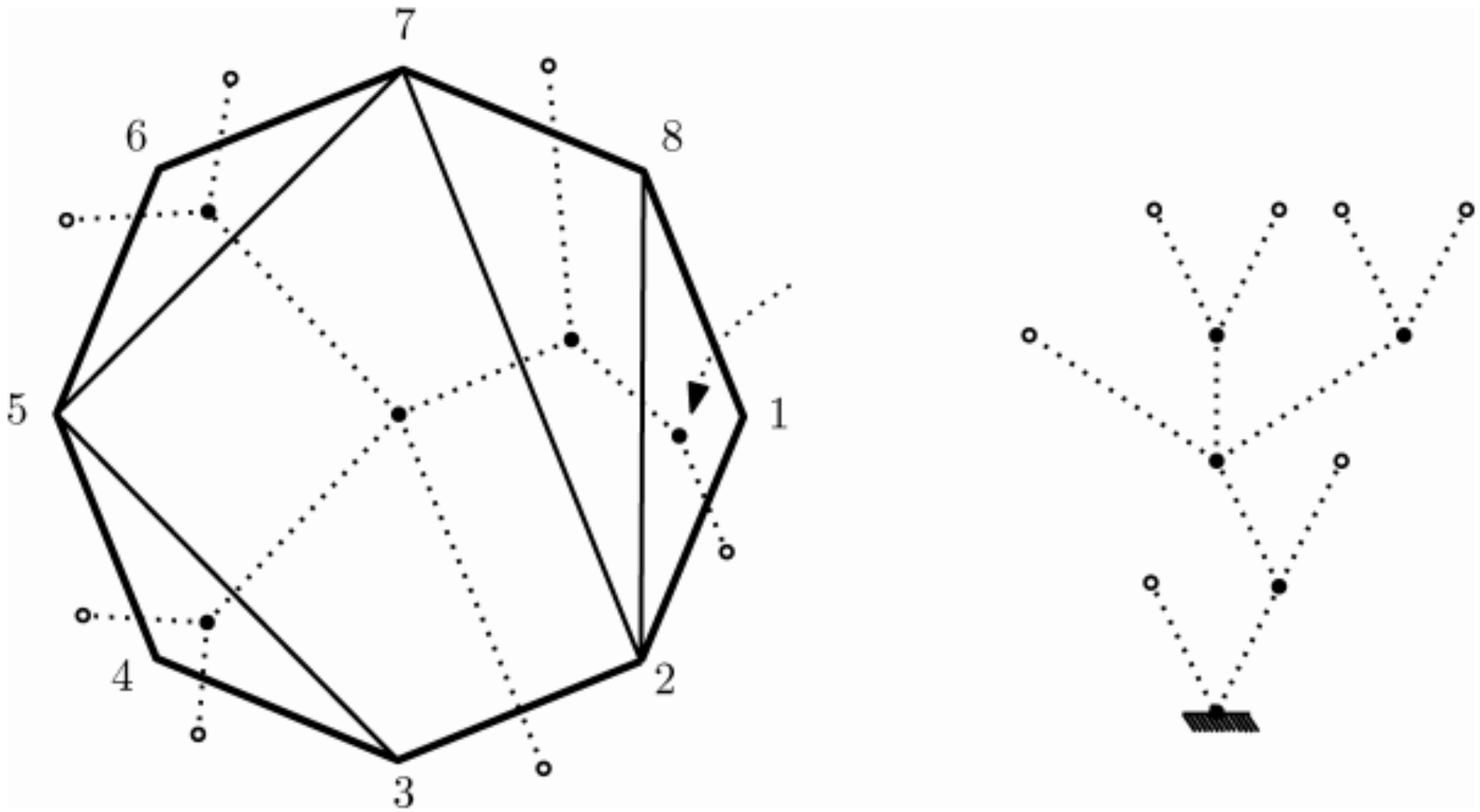


Fig. 4. The dual tree of a dissection of P_8, note that the tree has 7 leaves.

Maps (Schaeffer)

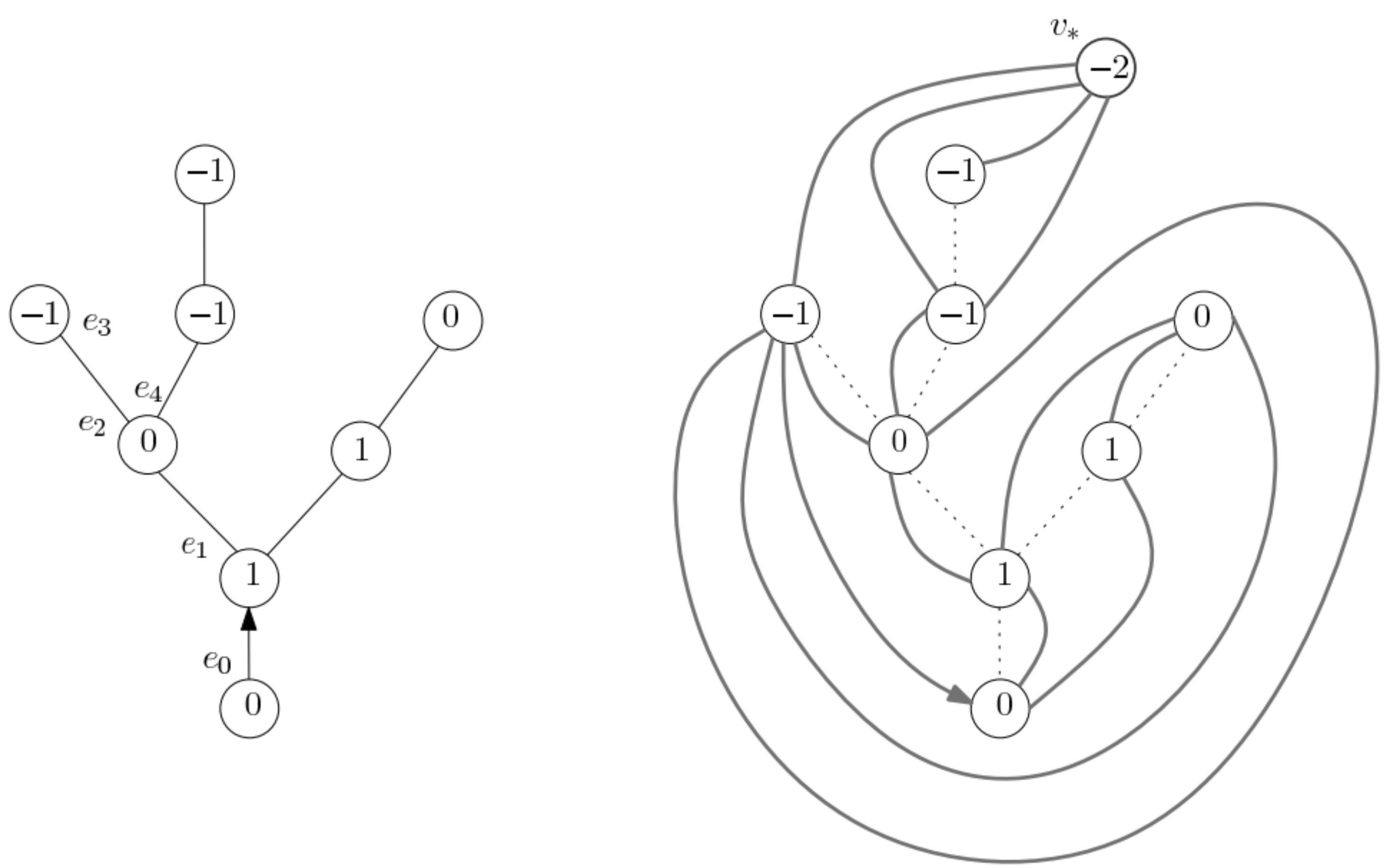
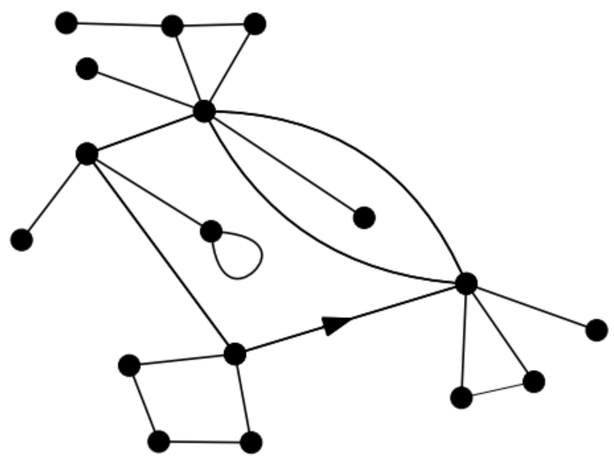
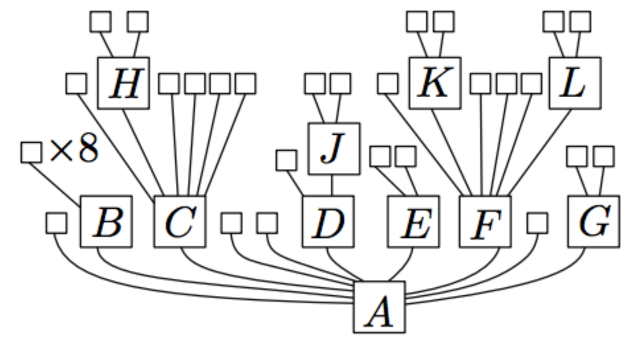


FIGURE 6. Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, e_3 is the successor of e_0, e_2 the successor of e_1, and so on.

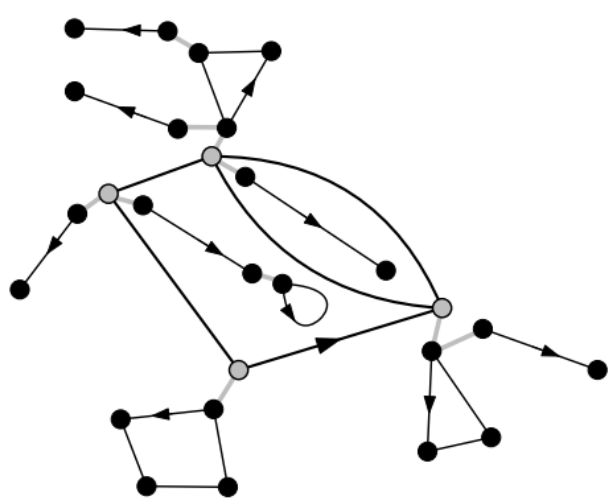
Maps (Addario-Berry)



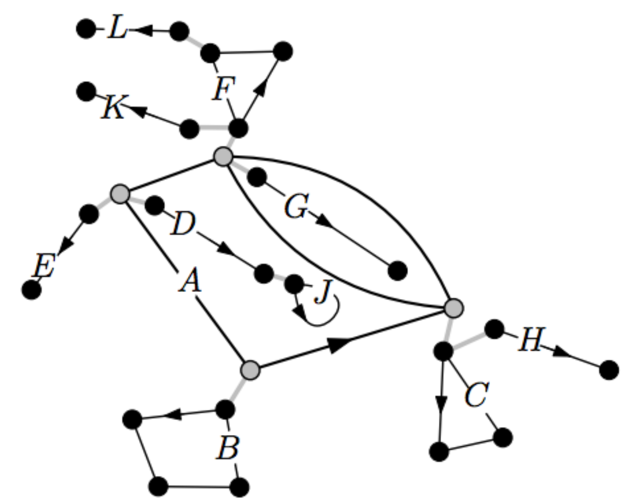
(A) A map M.



(B) The tree T_M. Tiny squares represent trivial blocks.

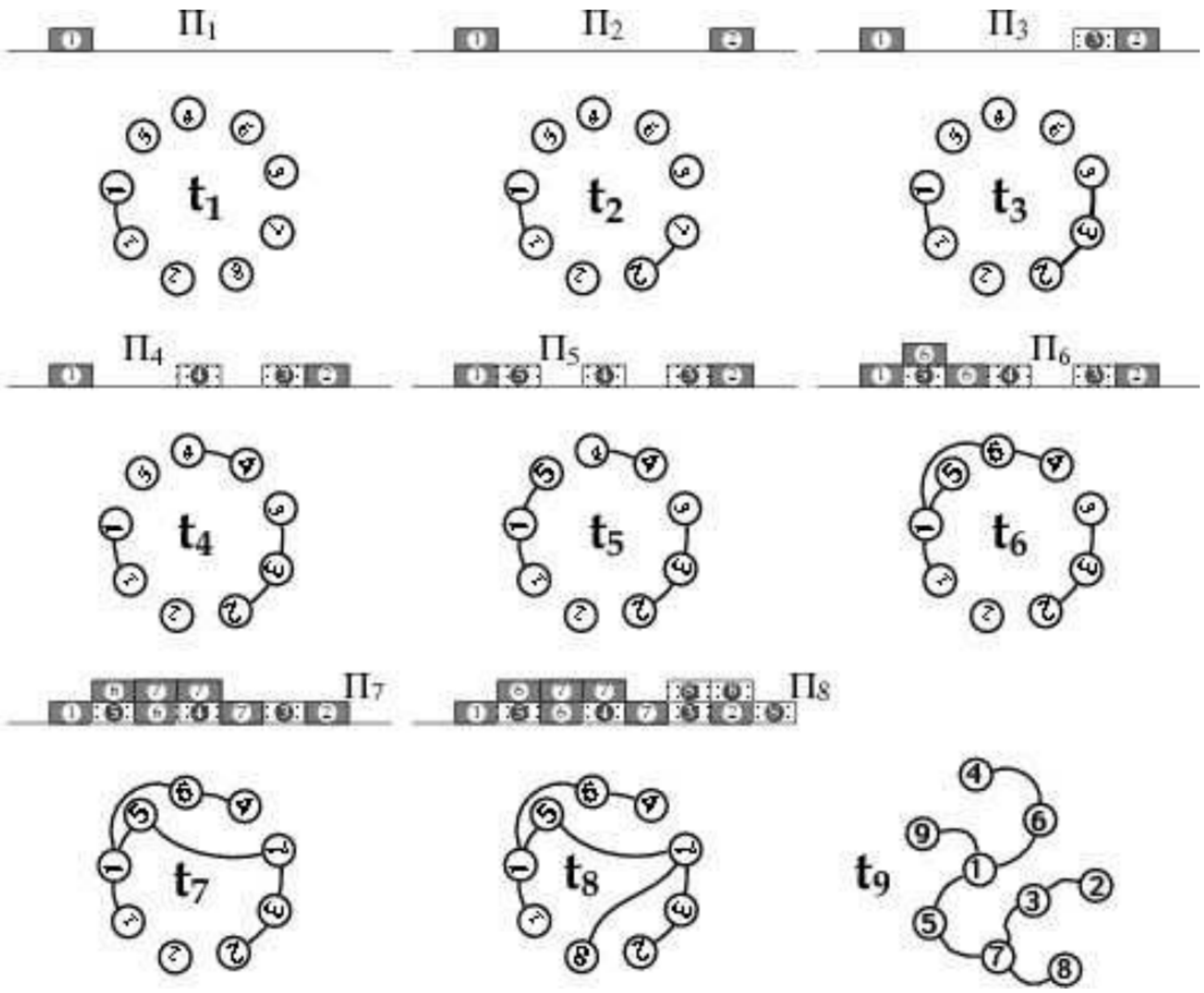


(C) The decomposition of M into blocks. Blocks are joined by grey lines according to the tree structure. Root edges of blocks are shown with arrows.



(D) The correspondence between blocks and nodes of T_M. Non-trivial blocks receive the alphabetical label (from A through L) of the corresponding node.

Parking functions (Chassaing, Louchard)



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II. Bienaymé trees

III. Local limits of Bienaymé trees

IV. Scaling limits of Bienaymé trees

Plane trees

Figure: Two different plane trees

We consider plane (i.e. rooted ordered) trees.

Bienaymé trees

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A random tree with law $\mathbb{P}_\mu$ is called a μ-Bienaymé tree (or B_μ tree).

Size-conditioned Bienaymé trees

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– when $\mu(k) = e^{-1}/k!$ for $k \geqslant 0$, $\mathcal{T}_n$ is a uniform labelled tree (a.k.a. Cayley tree) with n vertices (after forgetting the labels).

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Let ρ_μ be the radius of convergence of $G_\mu(z) = \sum \mu(i) z^i$.

⇝ If $c < \rho_\mu$, two Bienaymé trees with offspring distributions μ and μ_c, defined by

$$\mu_c(k) = \frac{1}{G_\mu(c)} c^k \mu(k), \qquad k \geqslant 0,$$

when conditioned on having n vertices, have the same distribution (Kennedy '75).

What are the limits of large size-conditioned Bienaymé trees?

What does a large size-conditioned Bienaymé tree look like, near the root?

Local limits

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Theorem (Kesten '87, Janson '12, Abraham & Delmas '14)

The convergence

$$\mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_\infty$$

holds in distribution for the local topology, where $\mathcal{T}_\infty$ *is the infinite Bienaymé tree conditioned to survive.*

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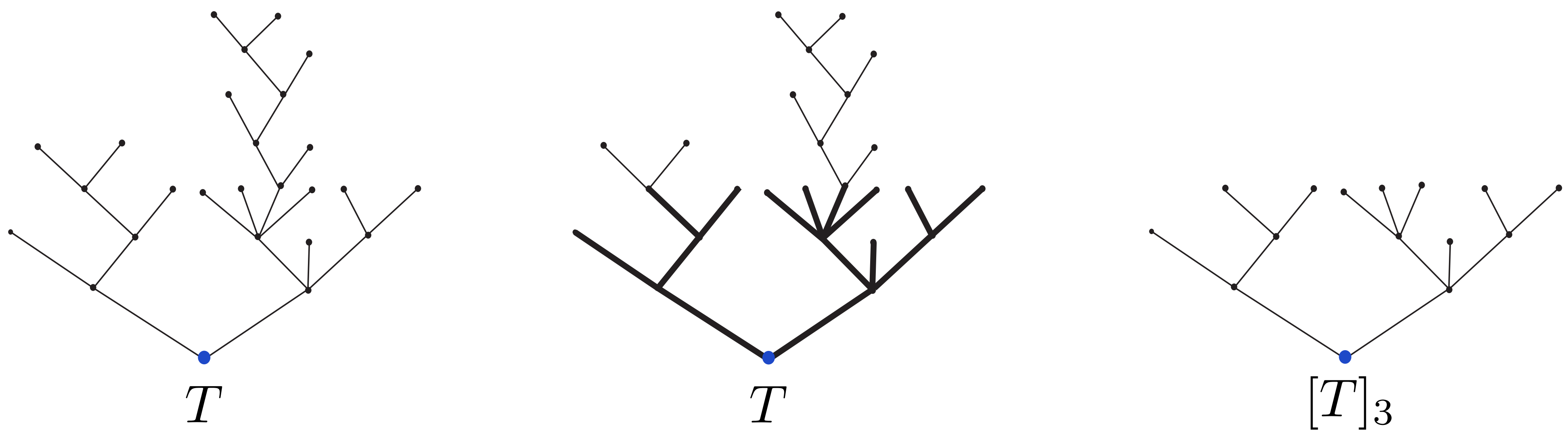
$\rightsquigarrow$ This means that $[\mathcal{T}_n]_k \to [\mathcal{T}_\infty]_k$ in distribution, where $[T]_k$ denotes the subtree of T obtained by keeping the first k generations:



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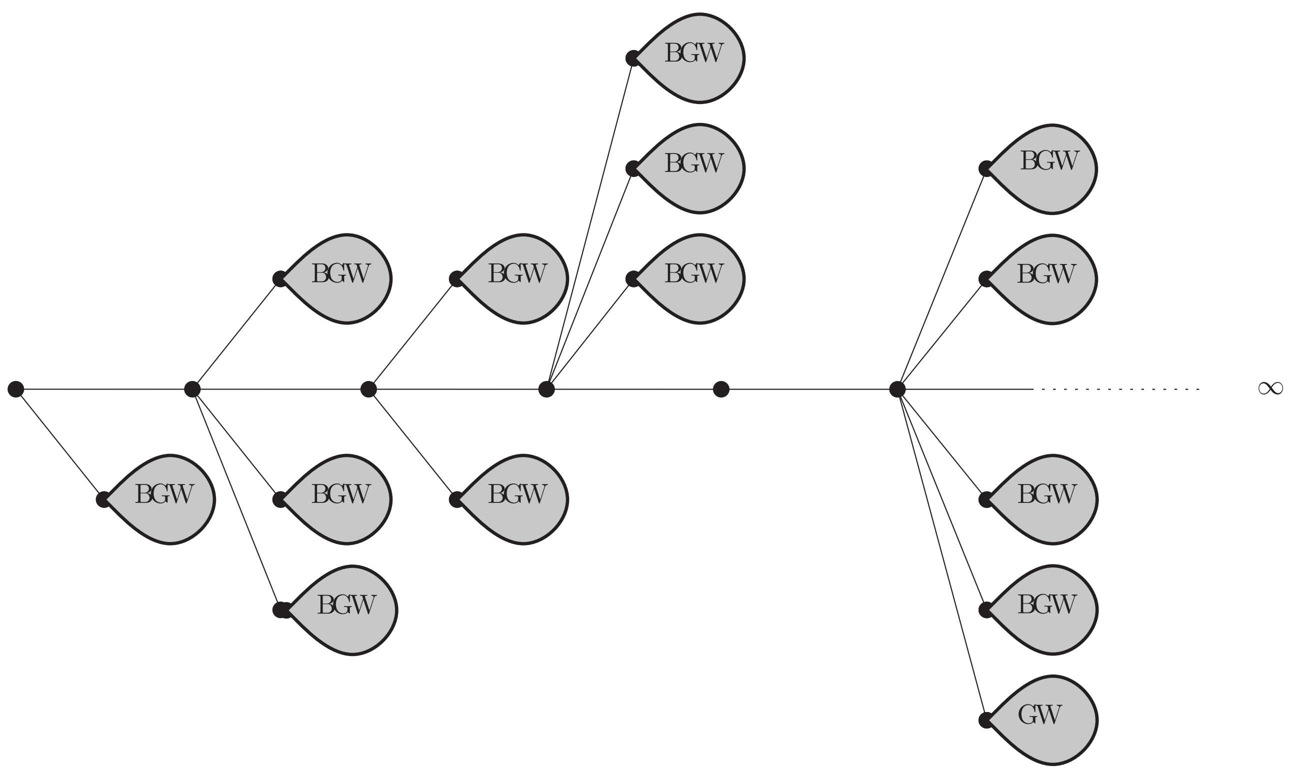

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Theorem (Jonsson & Stefánsson '11, Janson '12, Abraham & Delmas '14)

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$$\mathcal{T}_n \quad \xrightarrow[n \to \infty]{(d)} \quad \mathcal{T}^*_\infty$$

holds in distribution for the local topology, where $\mathcal{T}^*_\infty$ is a "condensation" tree

Local limits: subcritical case

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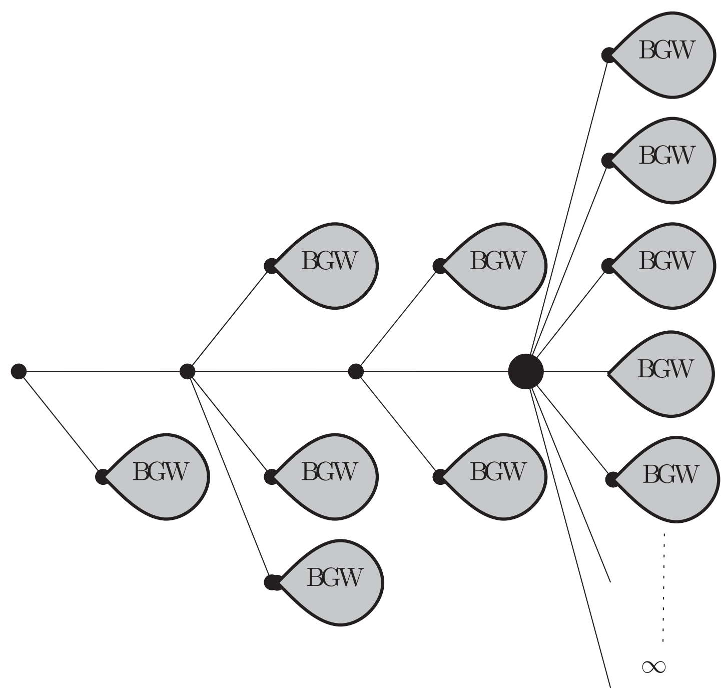

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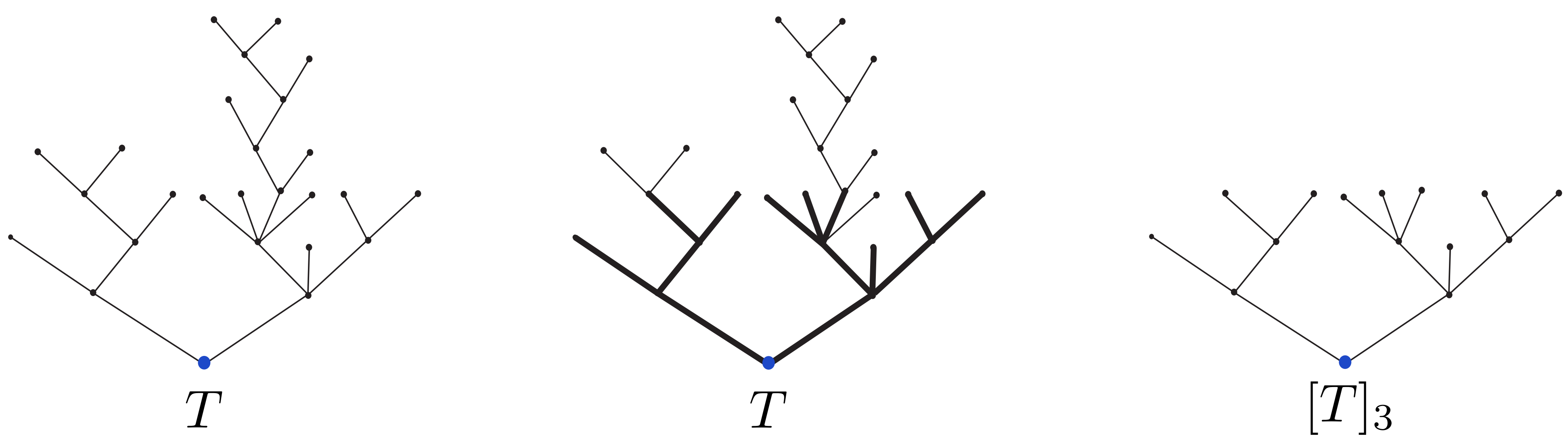
⇝ This means that $[\mathcal{T}_n]_k \to [\mathcal{T}_\infty]_k$ in distribution, where $[T]_k$ denotes the subtree of T obtained by keeping *the first* k *children* on the first k generations:



I. Models coded by trees

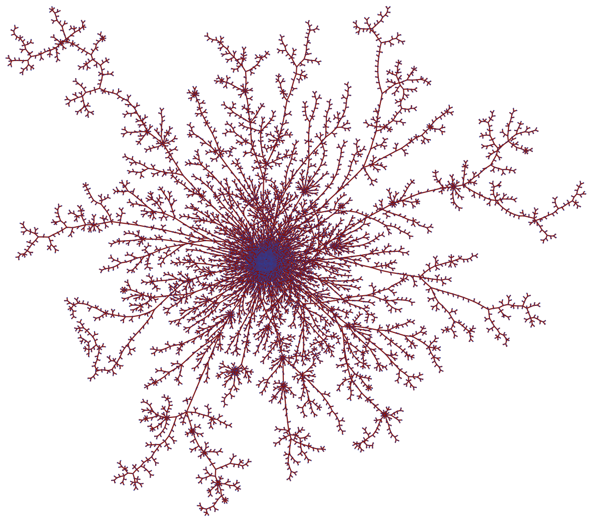
II. Bienaymé trees

III. Local limits of Bienaymé trees

IV. Scaling limits of Bienaymé trees

What does a large Bienaymé tree look like, globally?

I have simulated and drawn a uniform plane tree with 10000 vertices. What did I get?



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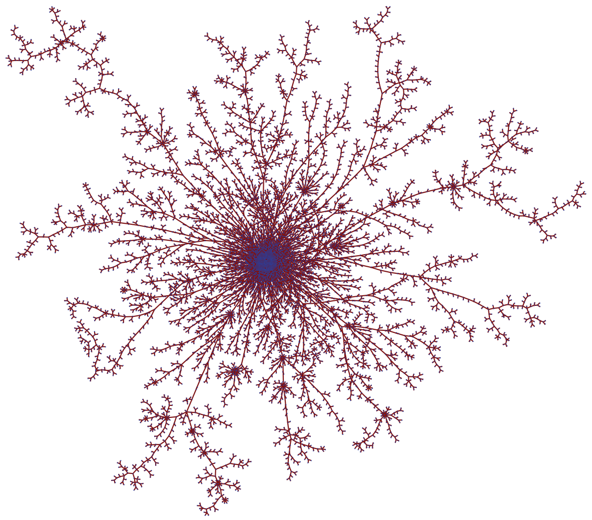

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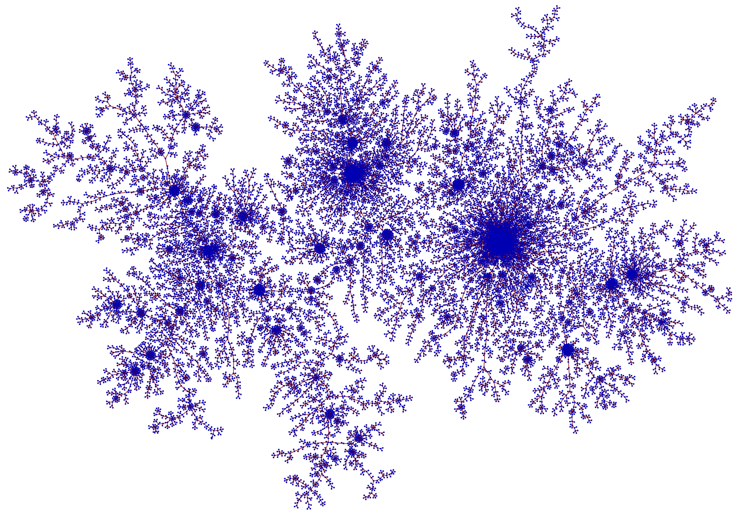

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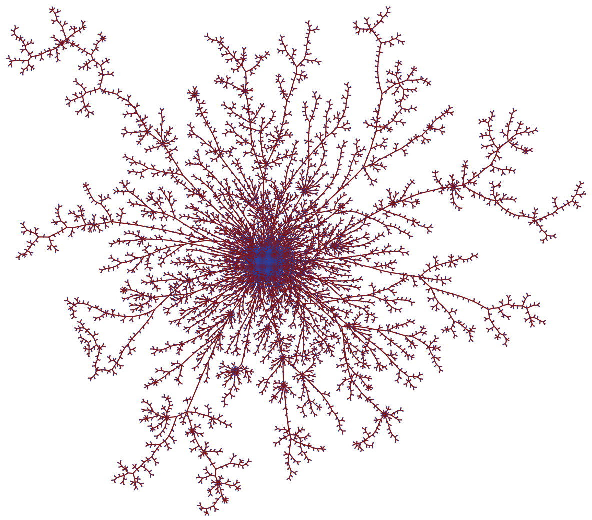

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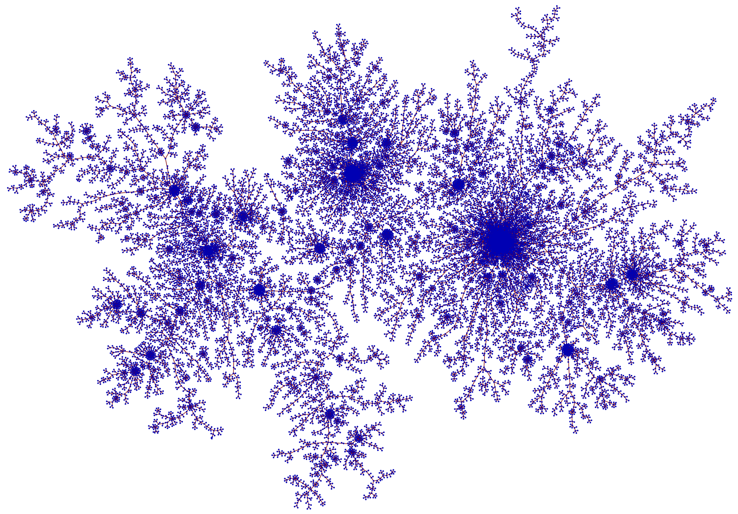

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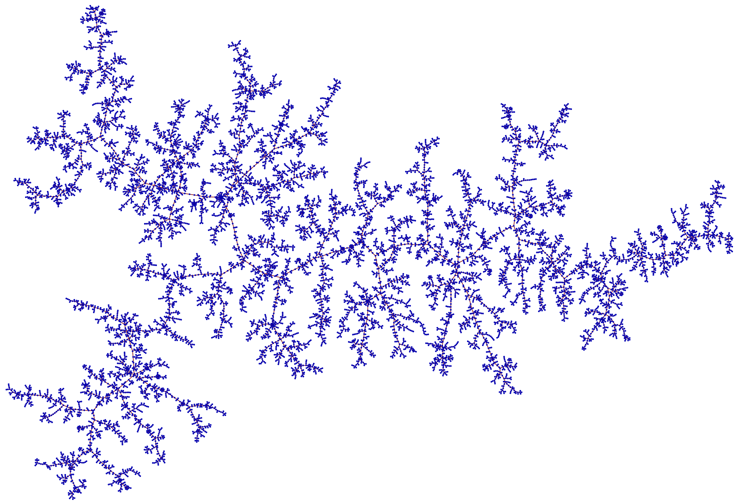

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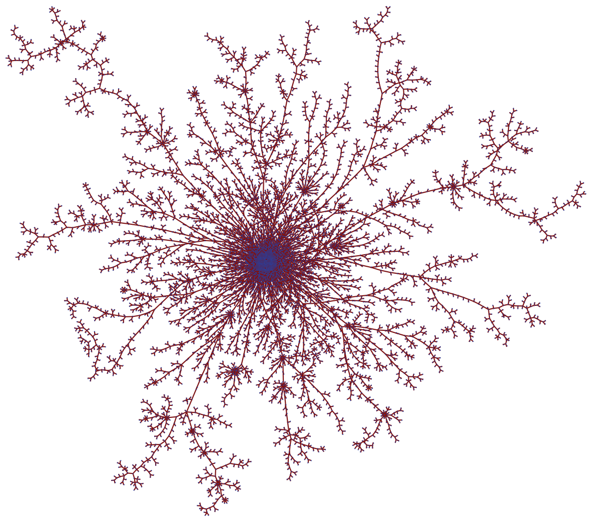

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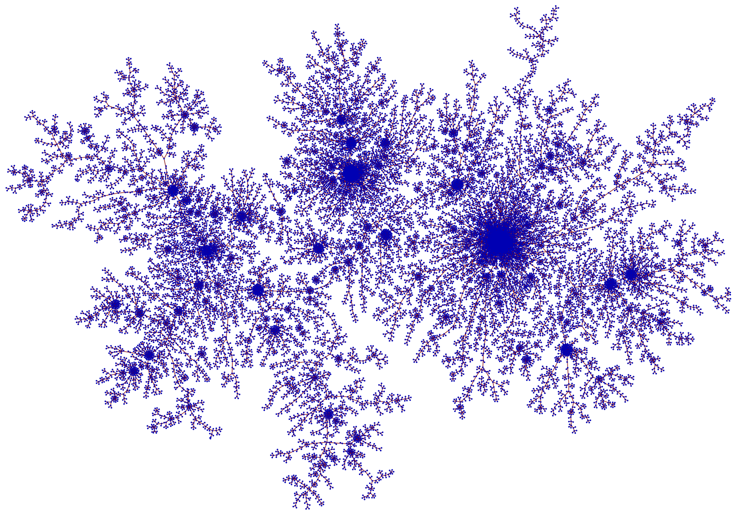

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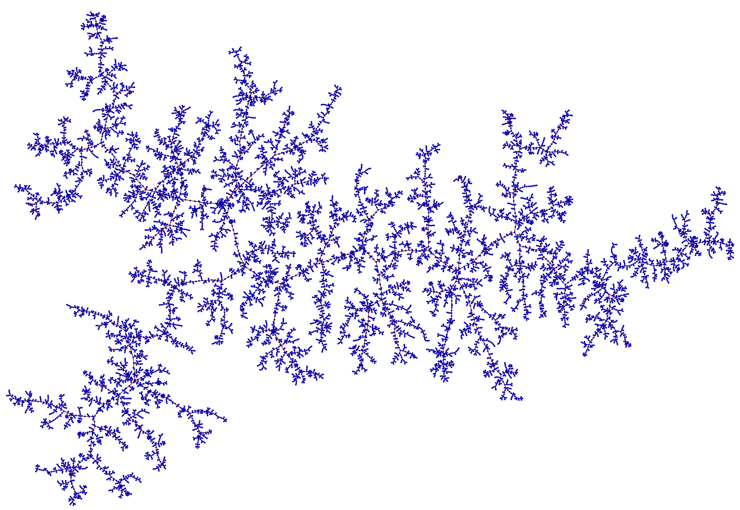

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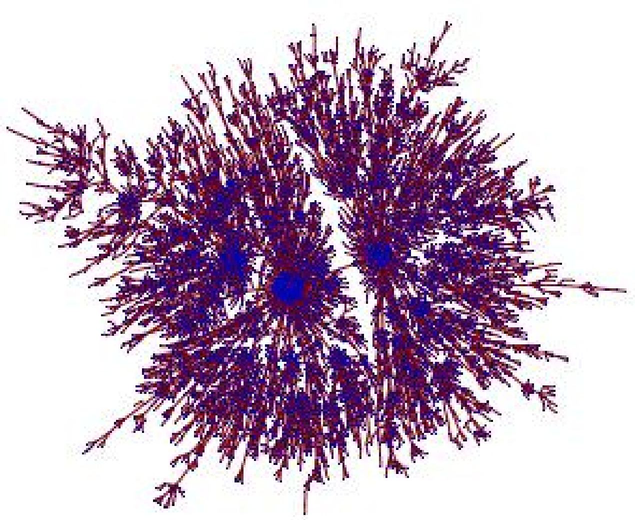

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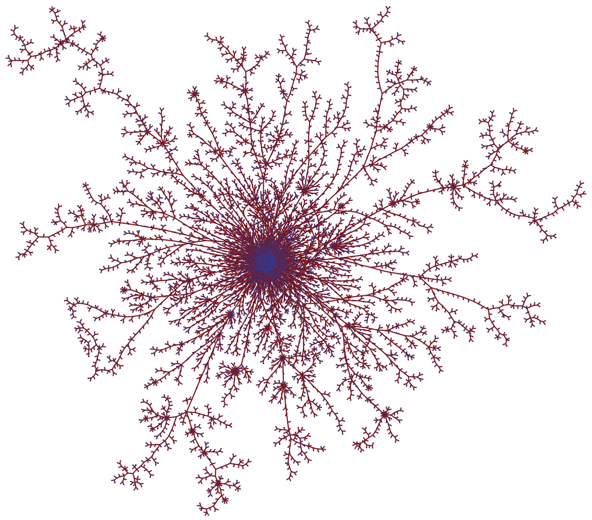

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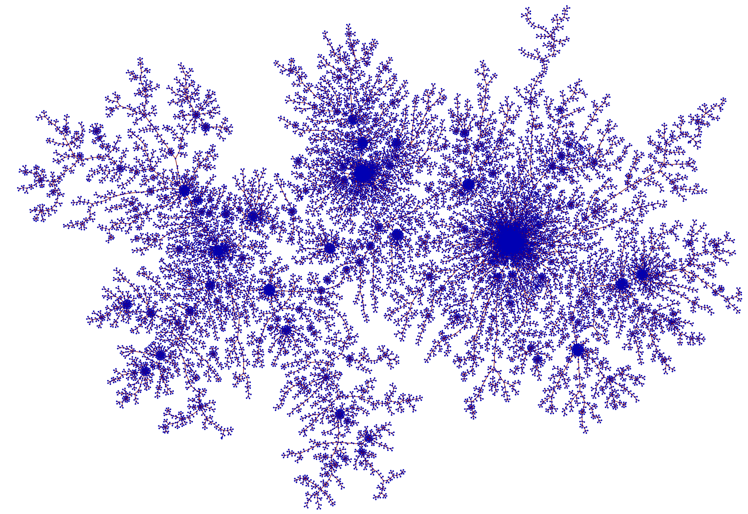

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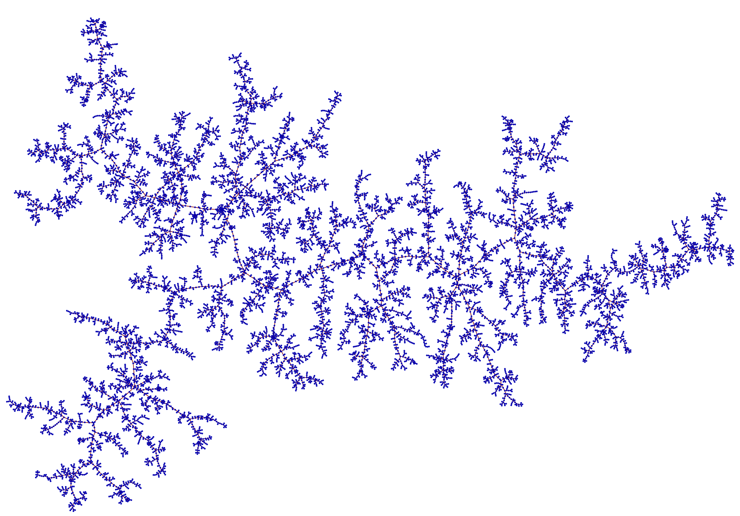

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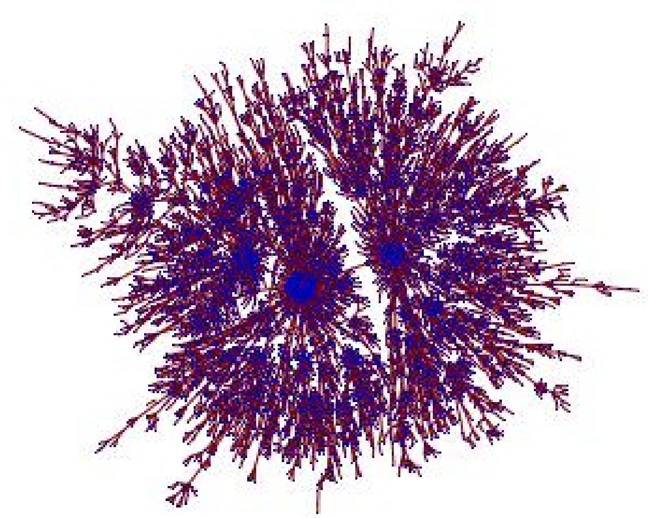

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$\rightsquigarrow$ wooclap.com ; code **randomtree**.

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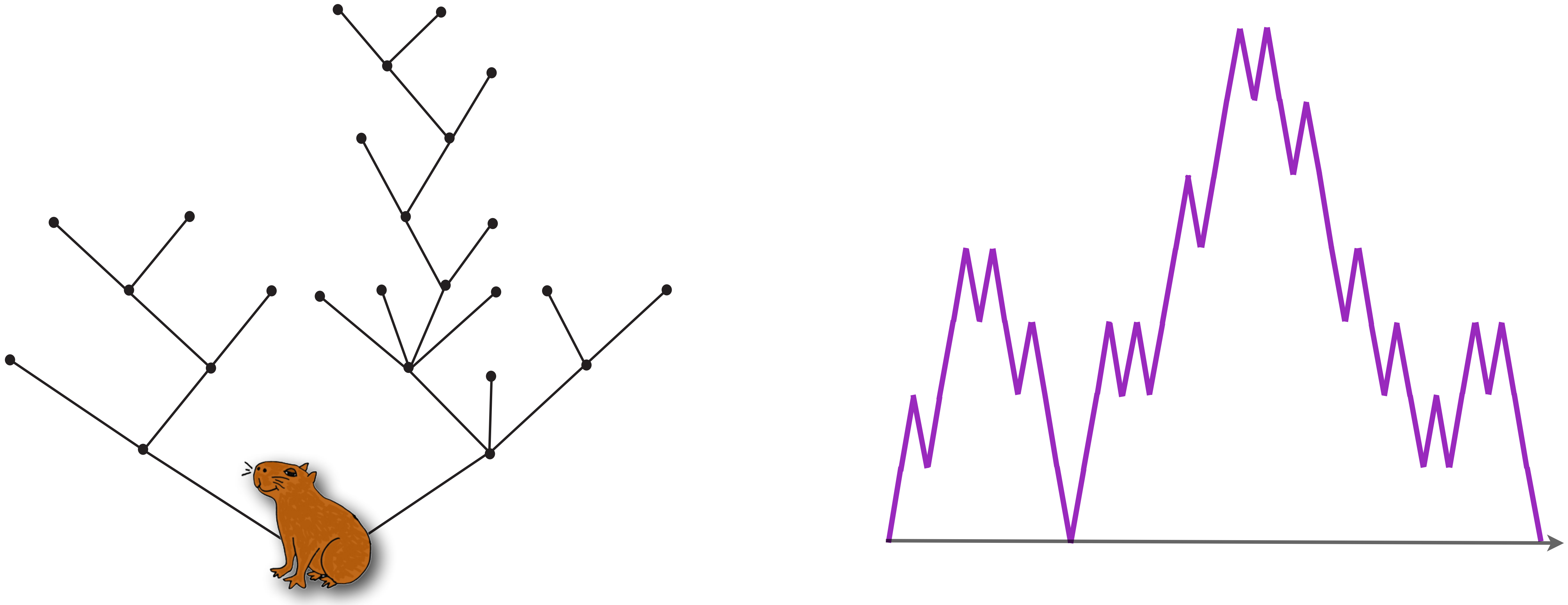
We shall code plane trees by functions.

Coding trees by functions

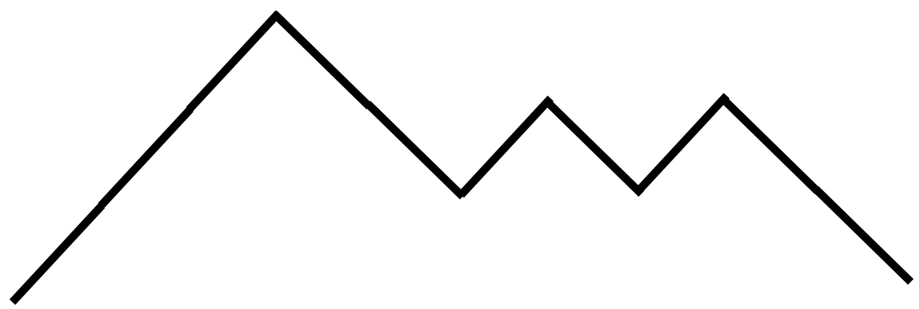
Contour function of a tree

Define the contour function of a tree:



Coding trees by contour functions

Knowing the contour function, it is easy to recover the tree.



Scaling limits

Scaling limits: finite variance

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Let σ^2 be the variance of μ. Let $t \mapsto C_t(\mathcal{T}_n)$ be the contour function of $\mathcal{T}_n$. Then:

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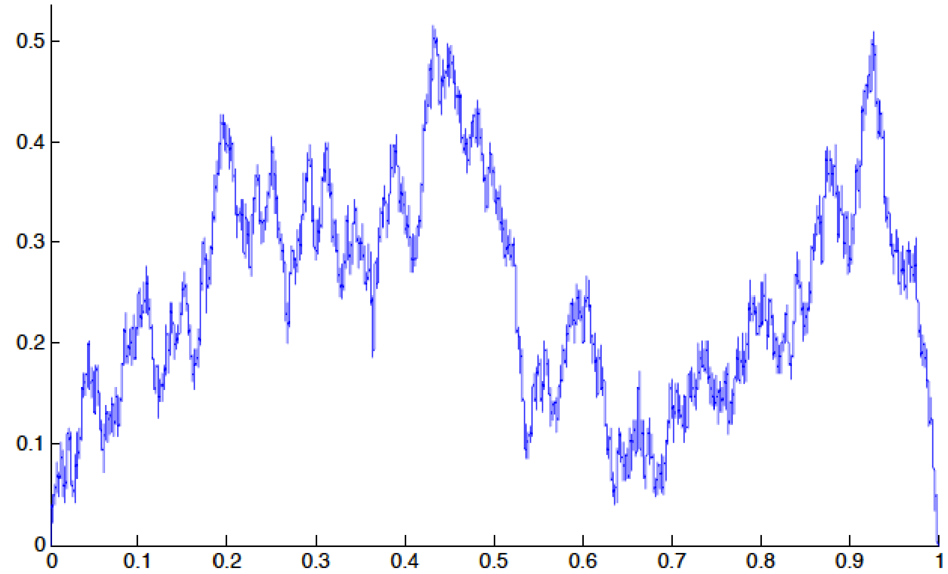

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Idea of the proof:

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$\leadsto$ conditioned Donsker's invariance principle.

Do the discrete trees converge to a continuous tree?

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Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!

The Hausdorff distance

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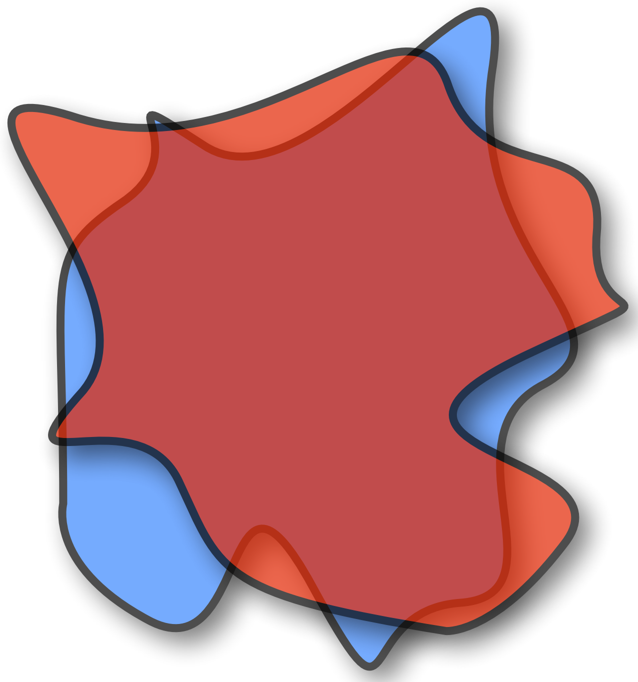

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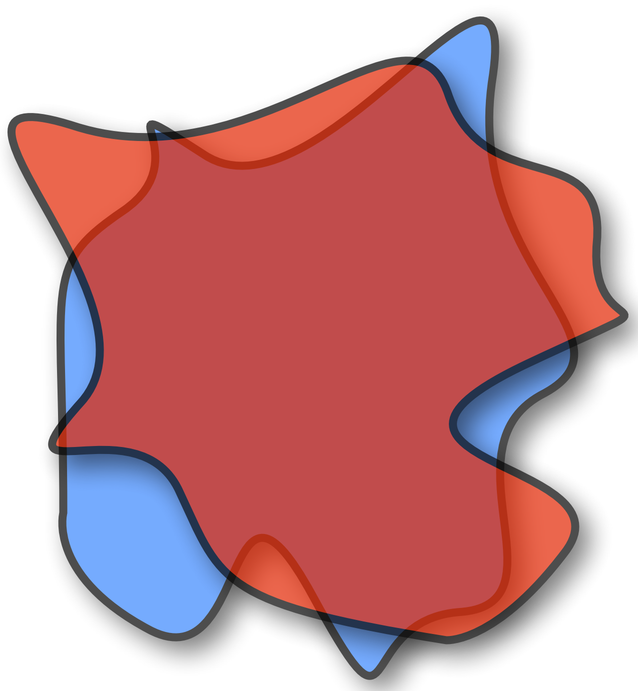

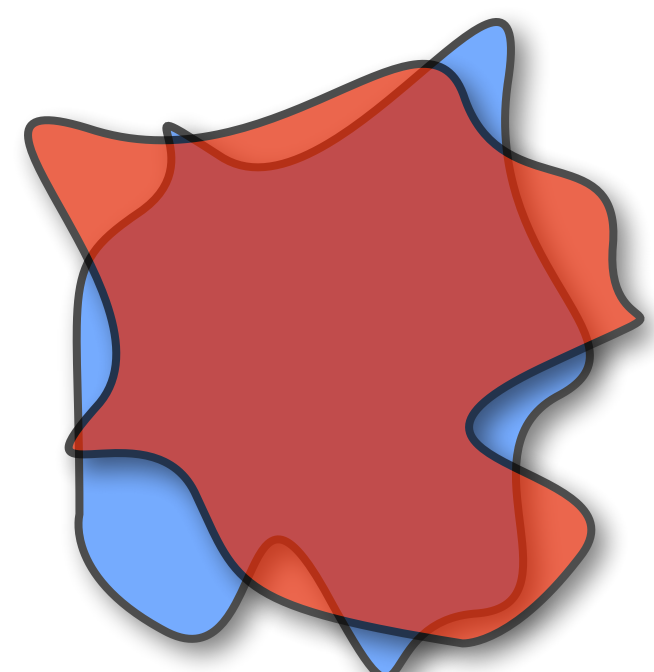

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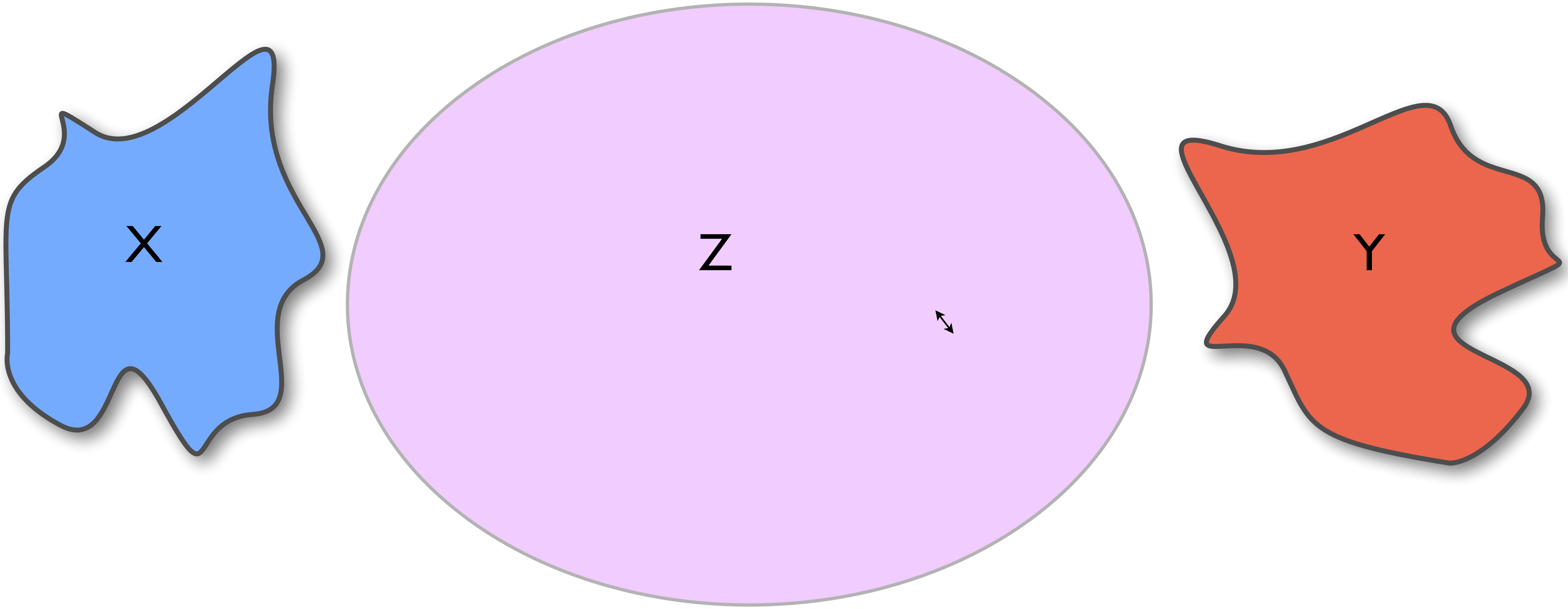

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y in a *same* metric space Z.

The Brownian tree

$\rightsquigarrow$ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

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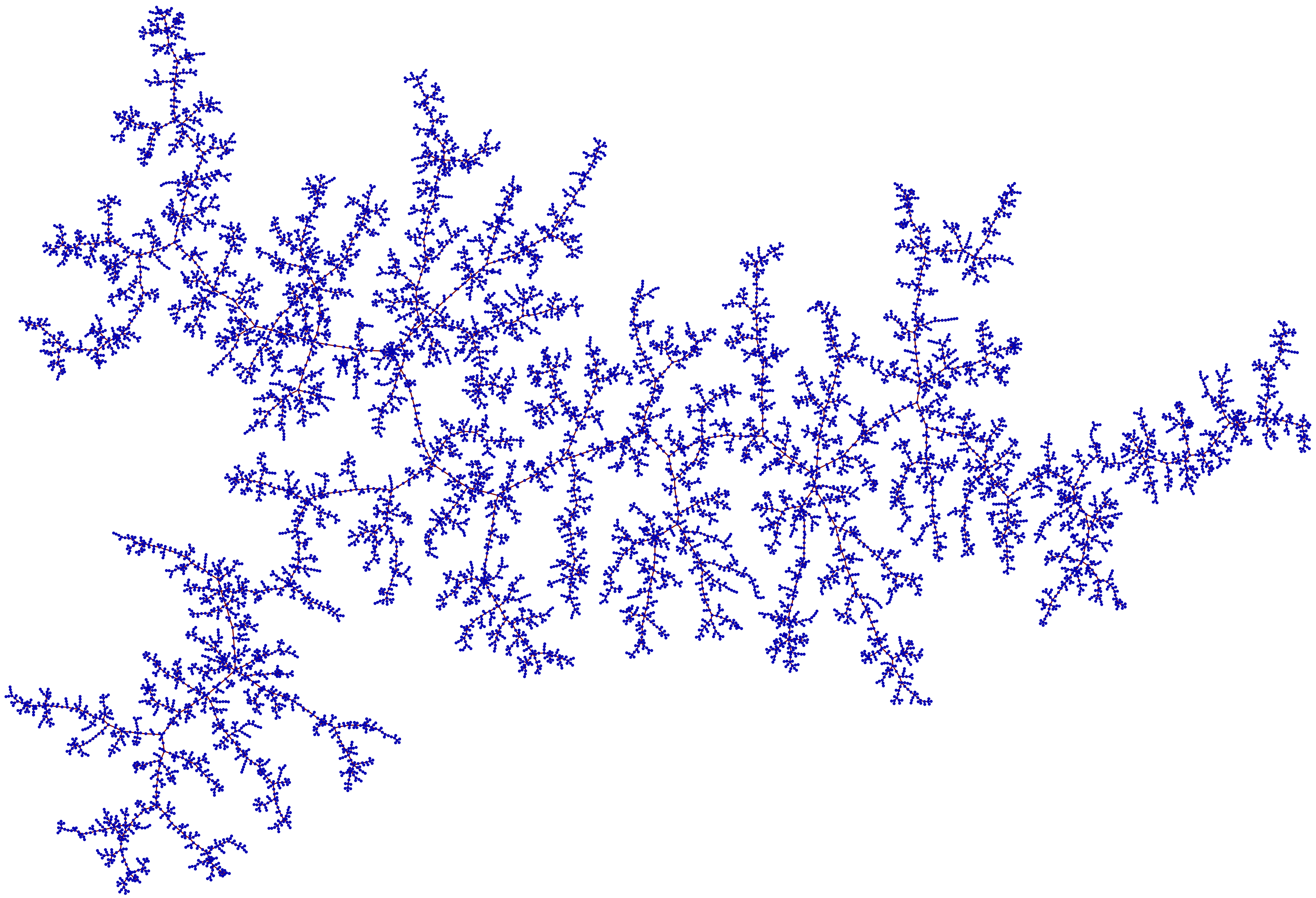
The metric space $\mathcal{T}_{\mathbb{e}}$ is called the *Brownian continuum random tree (CRT)*, and is coded by a Brownian excursion.



An approximation of a realization of a Brownian CRT

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⇝ Scaling limits are described use stable Lévy processes.

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⤳ Why did Aldous consider only critical offspring distributions?

WHAT ABOUT NON-CRITICAL OFFSPRING DISTRIBUTIONS?

↝ Why did Aldous consider only critical offspring distributions?

4. Because we condition on total population size, the distribution of $\mathscr{T}_n$ is unchanged by replacing ξ with another distribution χ in the same exponential family

$$P(\xi = i) = c\theta^i P(\chi = i), \qquad i \geq 0 \text{ for some } c, \theta.$$

Thus there is no essential loss of generality in considering only critical branching processes.

Condensation (subcritical case)

Let μ be a **subcritical** offspring distribution such that $\mu(n) \sim c/n^{1+\beta}$ with $\beta > 2$. Let $\mathcal{T}_n$ be a μ-Bienaymé tree conditioned on having n vertices.

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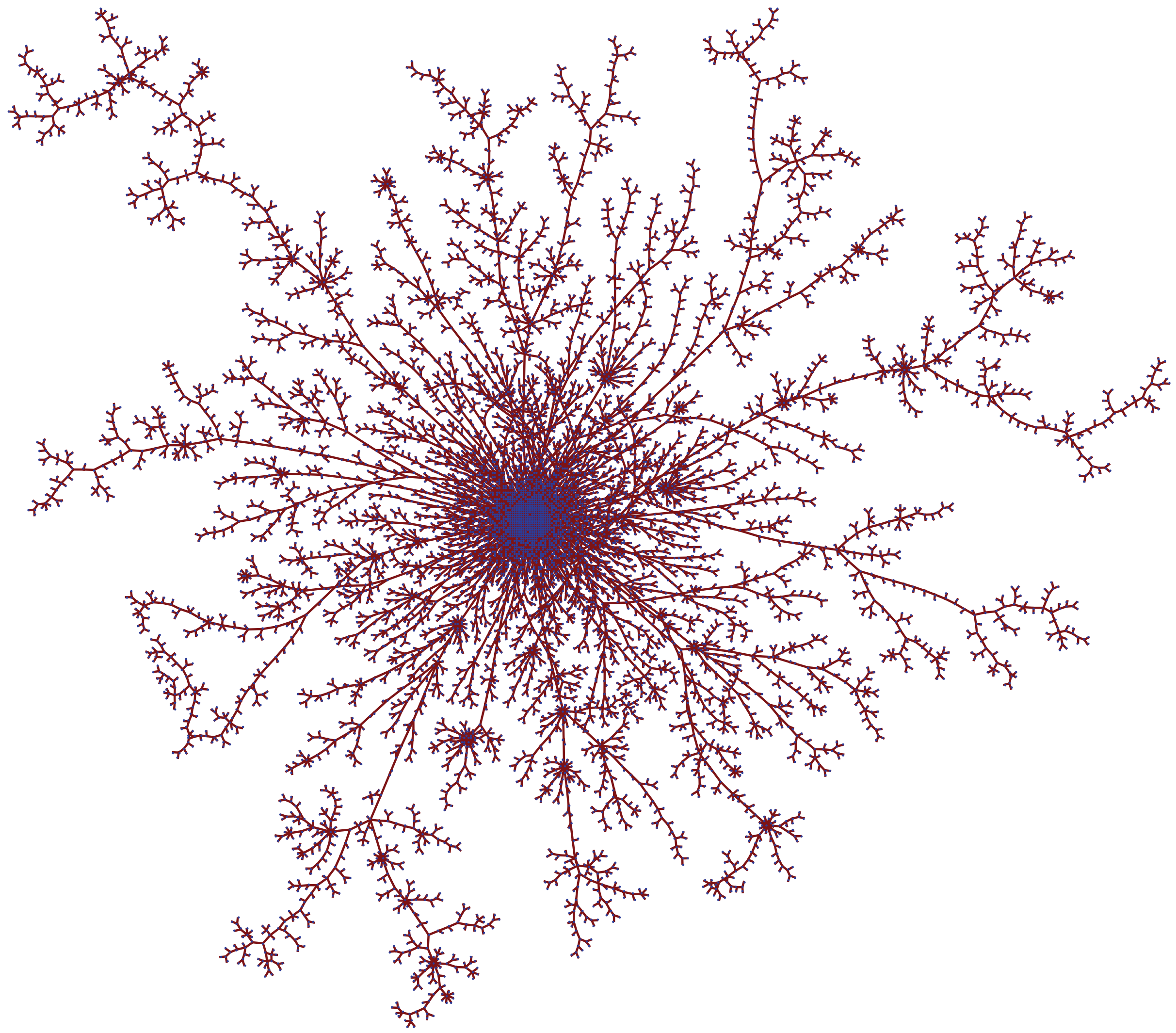

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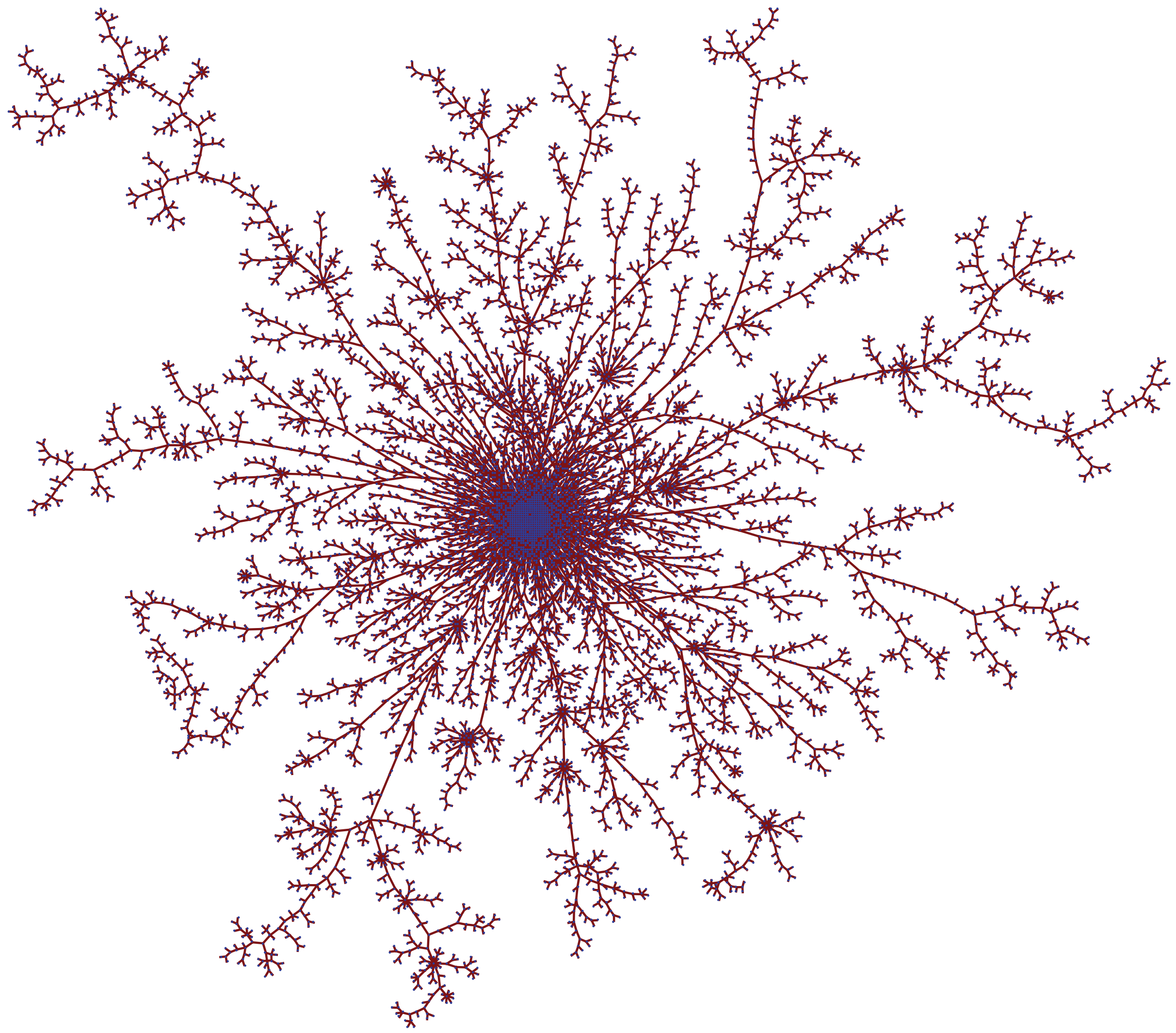

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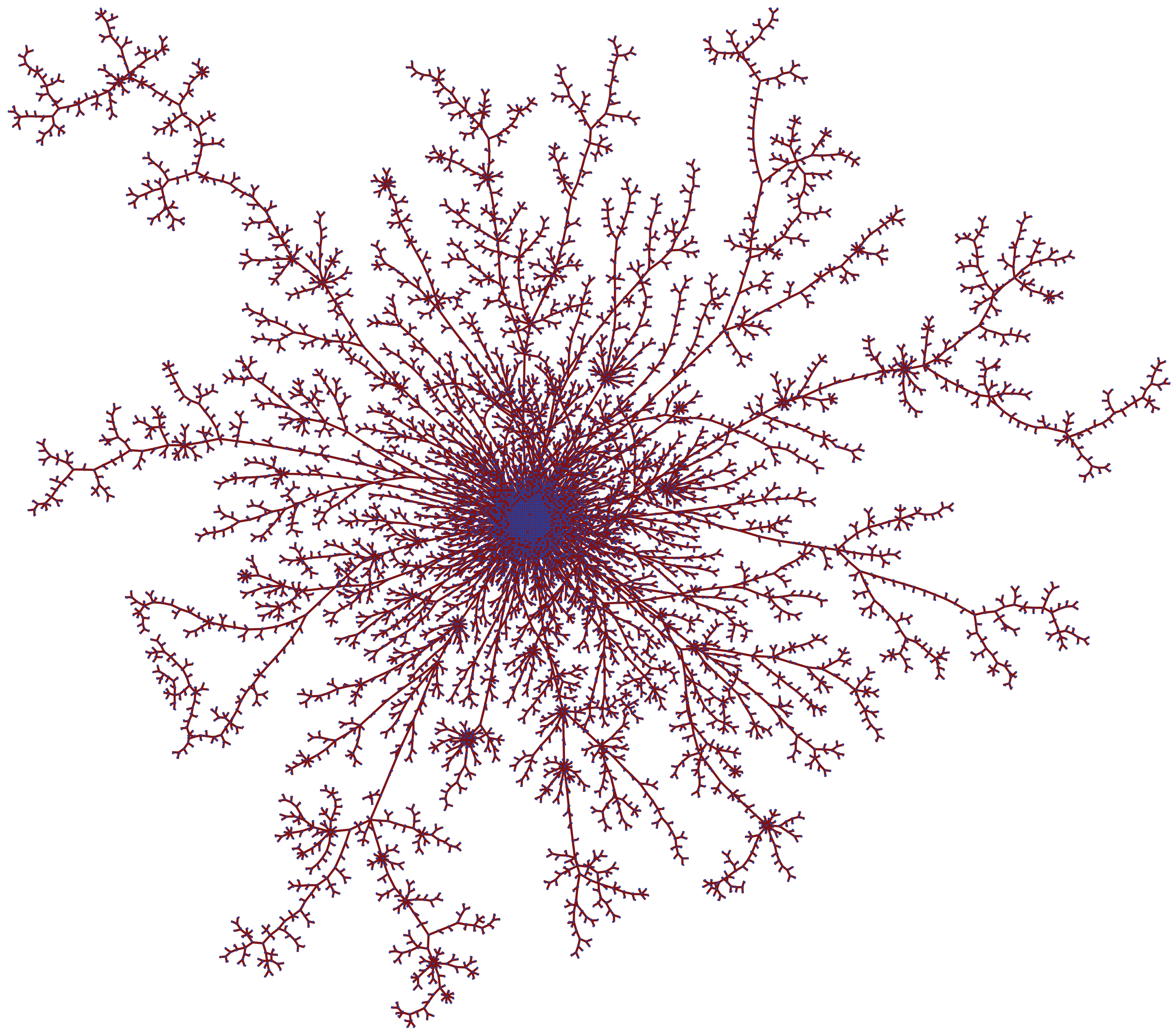

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- ⇝ This will be the focus of the mini-course.