3) An application

Corollary Set $D_n = \max(X_1, \dots, X_n)$ and let $D_n^{(2)}$ be the second largest element of $(X_1, \dots, X_n)$. Then:

① $\forall u \geq 0 \quad \mathbb{P}\left(\frac{D_n^{(2)}}{n^{1/\beta}} \leq u \,\Big|\, S_n \in [x_n, x_n+1) \right) \xrightarrow[n\to\infty]{} \exp\left(-\frac{c_0}{u^\beta}\right)$

② $\frac{D_n - x_n}{\sigma\sqrt{n}}$ under $\mathbb{P}(\cdot \mid S_n \in [x_n, x_n+1)) \xrightarrow{(d)} N(0,1)$ with $\sigma^2 = \mathrm{Var}(X_1)$

③ $\frac{D_n}{x_n}$ under $\mathbb{P}(\cdot \mid S_n \in [x_n, x_n+1)) \xrightarrow{\mathbb{P}} 1$

Proof ① By the theorem, it is enough to show that $\mathbb{P}\left(\frac{\max(X_1, \dots, X_{n-1})}{n^{1/\beta}} \leq u \right) \xrightarrow[n\to\infty]{} \exp\left(-\frac{c_0}{u^\beta}\right)$

For this, the probability is $(1 - \mathbb{P}(X_1 \geq u n^{\frac{1}{\beta}}))^{n-1}$

$$= \exp\left((n-1) \ln\left(1 - \mathbb{P}(X_1 \geq u n^{\frac{1}{\beta}})\right)\right)$$

But we have seen that $\mathbb{P}(X_1 \geq u n^{1/\beta}) \underset{n\to\infty}{\sim} \frac{c_0}{u^\beta n}$, and the result follows.

② Observe that $\frac{D_n - x_n}{\sigma\sqrt{n}} = \frac{S_n - x_n}{\sigma\sqrt{n}} - \frac{(\hat{X}_1 + \dots + \hat{X}_{n-1})}{\sigma\sqrt{n}}$

Under $\mathbb{P}(\cdot \mid S_n \in [x_n, x_n+1))$, $|S_n - x_n| \leq 1$, so $\frac{S_n - x_n}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} 0$ and $\frac{\hat{X}_1 + \dots + \hat{X}_{n-1}}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{(d)} N(0,1)$ by the central limit theorem. The result follows

③ Since $\frac{D_n}{x_n} - 1 = \frac{\sigma\sqrt{n}}{x_n} \cdot \frac{D_n - x_n}{\sigma\sqrt{n}}$ and $\frac{\sigma\sqrt{n}}{x_n} \xrightarrow[n\to\infty]{} 0$, this follows from ②.

III) Condensation phenomena in random trees

Outline: 1) Largest degrees
2) Structure of the tree
3) Height of the condensation vertex

Here our framework is the following:

$\mathcal{T}_n$ is a $\mathbb{P}_\mu$-tree conditioned on having n vertices with

① $m = \sum_{i \geq 0}^{\infty} i \mu(i) < 1$

② $\mu(n) \underset{n \to \infty}{\sim} \frac{c}{n^{1+\beta}}$ with $c>0, \beta > 2$.

The first study of such trees is quite recent (Jonsson & Stefansson '11)!

1) Largest degrees

Let $\Delta(\mathcal{T}_n)$ be the largest number of children in $\mathcal{T}_n$ and $\Delta^2(\mathcal{T}_n)$ the second largest number of children.

Theorem 1 (condensation)

① $\frac{\Delta(\mathcal{T}_n) - (1-m)n}{\sigma \sqrt{n}} \xrightarrow[n \to \infty]{(d)} \mathcal{N}(0,1)$ with $\sigma^2 = \mathrm{Var}(\mu)$

② $\forall u > 0$, $\mathbb{P}\left(\frac{\Delta^2(\mathcal{T}_n)}{n^{1/\beta}} \leq u \right) \xrightarrow[n \to \infty]{} \exp\left(- \frac{c}{\beta} \cdot \frac{1}{u^\beta} \right)$.

Proof Recall that $(X_i)_{i \geq 1}$ are iid with law $\mathbb{P}(X_1 = k) = \mu(k+1)$ for $k \geq 1$, $W_n = X_1 + \cdots + X_n$.
Since the largest and second largest jumps are invariant under cyclic shifts, it follows that $(\Delta(\mathcal{T}_n) - 1, \Delta^2(\mathcal{T}_n) - 1)$ has the same law as the largest and second largest jump of $(X_1, \ldots, X_n)$ under $\mathbb{P}(\cdot \mid W_n = -1)$.

Observe that $\mathbb{E}[X_1] = m-1$. Thus, setting $\overline{X}_i = X_i + 1 - m$, $\overline{W}_n = W_n + n(1-m)$, we have $W_n = -1 \iff \overline{W}_n = n(1-m) - 1$ and $\overline{X}_1$ is centered and satisfies $(\mathcal{H}_\Delta)$ with $T = [0,1)$: $\mathbb{P}(\overline{X}_1 \in [u, u+1)) \underset{u\to\infty}{\sim} \frac{c}{u^\beta}$.

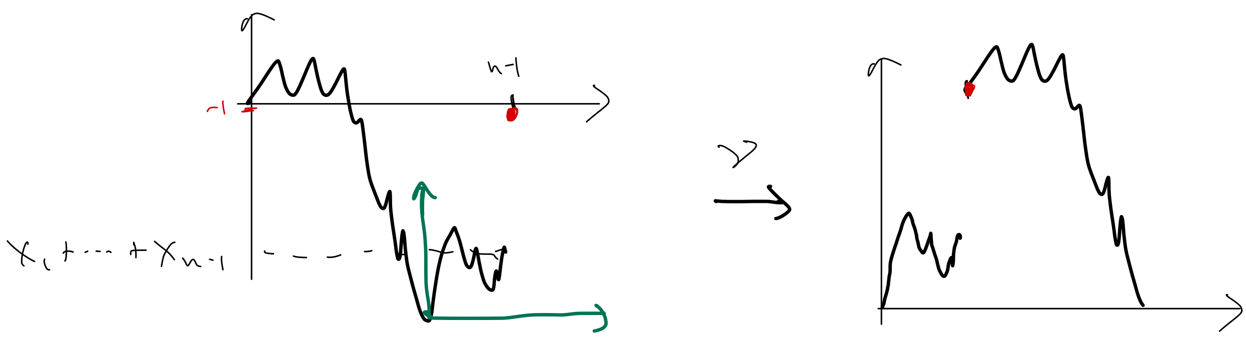
We can thus apply the one big-jump principle and its corollary and the desired result follows.

2) Structure of the tree

The one-big jump principle can actually allow to control the whole Łukasiewicz path of τ_n:

Theorem 2 We have $d_{TV}\left(\mathcal{W}(\tau_n), \text{walk with jumps } \mathcal{V}(X_1, \dots, X_{n-1}, -1-X_1-\dots-X_{n-1})\right) \xrightarrow[n\to\infty]{} 0$

Observe that $X_1+\dots+X_n$ is typically $\simeq -(1-m)n$. The theorem says that for large n $\mathcal{W}(\tau_n)$ "looks like" the Vervaat transform of the random walk $(W_k)_{0\le k\le n-1}$ to which we add an artificial last jump so that it ends at -1.



Proof An adaptation of the proof of the one-big jump principle shows that
$d_{TV}\left((X_1,\dots,X_n) \text{ under } \mathbb{P}(\cdot \mid W_n=-1), (X_1,\dots,X_{U_n-1}, -1-X_1-\dots-X_{n-1}, X_{U_n+1},\dots,X_n)\right) \xrightarrow[n\to\infty]{} 0$
with U_n uniform on $\{1,\dots,n\}$ independent of $(X_1,\dots,X_{n-1})$.

Thus

$d_{TV}\left(\mathcal{V}(X_1,\dots,X_n) \text{ under } \mathbb{P}(\cdot \mid W_n=-1), \mathcal{V}(X_1,\dots,X_{U_n-1},-1-X_1-\dots-X_{n-1},X_{U_n+1},\dots,X_n)\right) \xrightarrow[n\to\infty]{} 0$

But $\mathcal{V}(X_1,\dots,X_{U_n-1},-1-X_1-\dots-X_{n-1},X_{U_n+1},\dots,X_n) = \mathcal{V}(X_{U_n+1},\dots,X_n,\dots,X_1,\dots,X_{U_n-1},-1-X_1-\dots-X_{n-1})$

and since U_n is uniform $\perp\!\!\!\perp (X_1,\dots,X_{n-1})$ we have

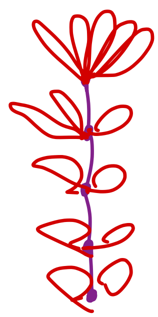
$$\nu(X_{U_n+1},\dots,X_n,\dots,X_1,\dots,X_{U_n-1},-1-X_1-\dots-X_{n-1}) \overset{(d)}{=} (X_1,\dots,X_{U_n-1},-1-X_1-\dots-X_{n-1},X_{U_n+1},\dots,X_n)$$

The conclusion follows from the fact that $\mathcal{W}(\mathcal{T}_n)$ has the same law as $\nu(X_1,\dots,X_n)$ under $\mathbb{P}(\cdot \mid W_n=-1)$

This can be used to study further properties of $\mathcal{T}_n$, e.g. to show:

$$\frac{\mathrm{Height}(\mathcal{T}_n)}{\ln(n)} \xrightarrow{\mathbb{P}} \ln(1/m)$$

Intuition $\mathcal{T}_n$ "looks like":



purple: spine with geometric length.

We graft $\perp\!\!\!\perp$ B_μ trees on the spine (in a certain way) and $\simeq (1-m)n$ $\perp\!\!\!\perp$ B_μ trees on the top of the spine.

3) Height of the condensation vertex

In this direction we show the following result

Theorem 3 Let $u_\star(\mathcal{T}_n)$ be the vertex with maximal degree of $\mathcal{T}_n$ (first in lexicographical order, if several)
Then for every $k \geq 0$, $\mathbb{P}(\mathrm{Height}(u_\star(\mathcal{T}_n))=k) \xrightarrow[n\to\infty]{} (1-m)\,m^k$

We will use the following result for random walks (proved later).
Recall that $\mathbb{E}[X_1]=m-1<0$, so $(W_n)_{n\geq 0}$ has a negative drift.

Proposition 1

1. $\mathbb{P}(W_i<0 \text{ for every } i\geq 1) = 1-m$
2. For every $k\geq 0$, $\mathbb{P}(\#\{n\geq 1 : W_n=\max(W_0,\dots,W_n)\}=k) = (1-m)\,m^k$

(observe that 2 follows from 1 by the strong Markov property)

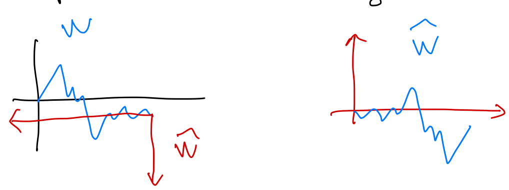
The proof uses several intermediate results

Lemma 1 For $1 \leq k \leq n$, $\mathbb{P}(W_1 < 0, \dots, W_n < 0, W_n = -k) = \frac{k}{n} \mathbb{P}(W_n = -k)$

For this we will use the following result, which is extension of the cycle lemma:

Lemma 2 For every $1 \leq k \leq n$, $\mathbb{P}(\inf\{i \geq 1 : W_i = -k\} = n) = \frac{k}{n} \mathbb{P}(W_n = -k)$

Proof of Lemma 1 The idea is to use the "time-reversed" random walk defined by:
$\hat{W}_k = W_n - W_{n-k}$ for $0 \leq k \leq n$ (we read the jumps backward)
$\hat{W}$ depends on n but we drop the dependence in n to simplify notation.
Geometrically this corresponds to considering the random walk from right to left, and rotating it:



The key property is that $(W_0, \dots, W_n) \overset{(d)}{=} (\hat{W}_0, \dots, \hat{W}_n)$ $(*)$
Then observe that

$$\{W_1 < 0, \dots, W_n < 0, W_n = -k\} = \{\inf\{i \geq 1 : \hat{W}_i = -k\} = n\}$$

Thus $\mathbb{P}(W_1 < 0, \dots, W_n < 0, W_n = -k) = \frac{k}{n} \mathbb{P}(\hat{W}_n = -k)$ (Lemma 2)
$= \frac{k}{n} \mathbb{P}(W_n = -k)$ ($(*)$)

Proof of Proposition 1

② follows from ① by the strong Markov property, so let us show ①.

Write $\mathbb{P}(W_i < 0 \text{ for every } i \geq 1) = \lim_{n\to\infty} \mathbb{P}(W_i < 0 \text{ for every } 1 \leq i \leq n)$
$= \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(W_i < 0 \text{ for every } 1 \leq i \leq n, W_n = -k)$
$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n} \mathbb{P}(W_n = -k)$

$$= \lim_{n\to\infty} \mathbb{E}\left[\frac{-W_n}{n} \mathbb{1}_{W_n<0}\right]$$

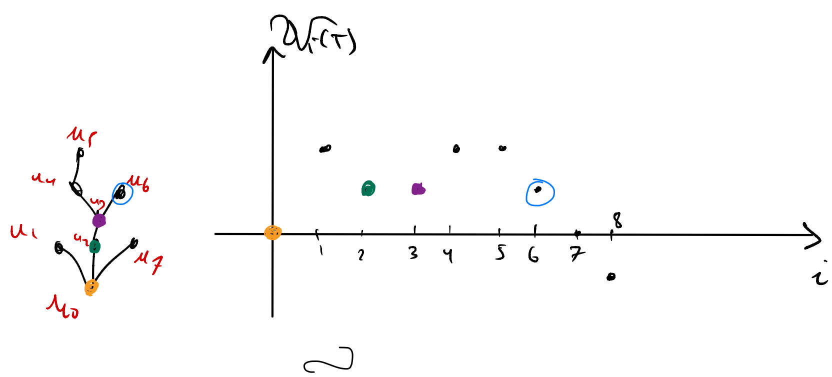
But $-\frac{W_n}{n} \mathbb{1}_{W_n<0} \xrightarrow[n\to\infty]{a.s.} -(m-1) = 1-m$ and $\left|\frac{-W_n}{n} \mathbb{1}_{W_n<0}\right| \leq 1$, so we get the desired result by dominated convergence ∽

We finally need the following result, which explains how to get heights of vertices from the Lukasiewicz path:

Proposition 2 Let T be a tree with n vertices $u_0 < u_1 < \cdots < u_{n-1}$ ordered in depth first search order. Then for every $0 \leq k \leq n$,

$$\text{Height}(u_k) = \#\left\{0 \leq j \leq k-1 : \mathcal{W}_j(T) = \min_{[\![j,k]\!]} \mathcal{W}(T)\right\}$$

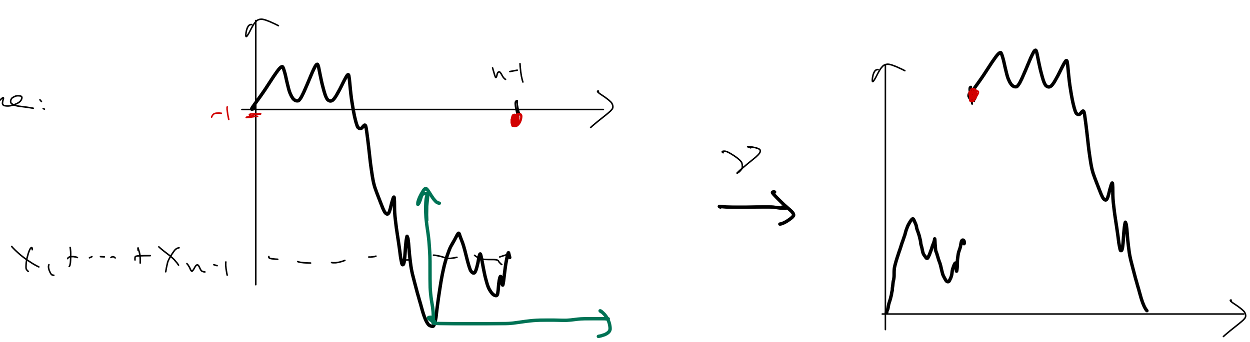
Proof This follows from the fact that elements of $\left\{0 \leq j \leq k-1 : \mathcal{W}_j(T) = \min_{[\![j,k]\!]} \mathcal{W}(T)\right\}$ correspond to vertices on the path from the root of T to u_k. For example:



∽

Proof of Theorem 3 By Theorem 2, it is enough to show the result with τ_n replaced with τ_n', the tree whose Lukasiewicz walk is the walk with jumps $\mathcal{V}(X_1, \ldots, X_{n-1}, -1-X_1-\cdots-X_{n-1})$

Recall the picture:



With probability tending to 1 as $n \to \infty$ the largest of the path $(0, W_1, \dots, W_{n-1}, -1)$ is the last one. Thus, by properties of the Vervaat transform and by Proposition 2, the height of the vertex with maximal degree (the one in "red" in the figure) is equal to

$$\#\{0 \le k \le n-2 : W_k = \inf_{[\![k, n-1]\!]} W\} = \#\{1 \le k \le n-1 : \hat{W}_k = \max_{[\![0,k]\!]} \hat{W}\}$$

where here $\hat{W}$ denotes the random walk time-reversed at time $n-1$: $\hat{W}_k = W_{n-1} - W_{n-1-k}$ for $0 \le k \le n-1$. Thus

$$\mathbb{P}(\mathrm{Height}(u_\star(\tau_n)) = j) \xrightarrow[n\to\infty]{} \mathbb{P}(\#\{k \geq 1 : \hat{W}_k = \max_{[\![0,k]\!]} \hat{W}\} = j)$$

and the desired result follows from Proposition 1

$\sim$