I | Bienaymé trees and random walks

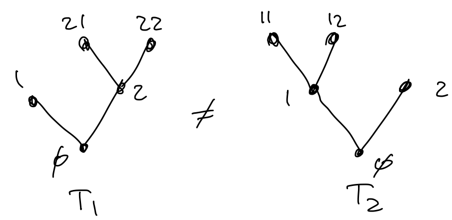
Outline: 1) Coding trees
2) Connection with conditioned random walks
3) The Vervaat transform.

1) Coding trees

Recall that we work with plane trees, for example:

$T_1 = \{\phi, 1, 2, 21, 22\}$

$T_2 = \{\phi, 1, 2, 11, 12\}$



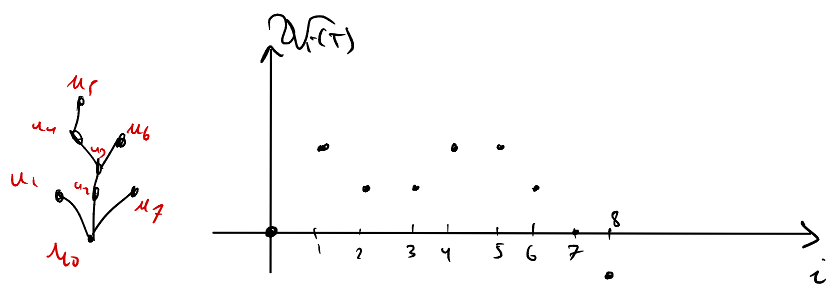
Formally, they can be defined as certain sets of labels (sequences of integers), we skip the formal definition
Informally, a plane tree can be seen as a genealogical tree where individuals are the vertices

Vertices of a plane tree can be equiped with the depth-first search order (informally, label vertices as soon as possible when doing the "contour" of the tree from left to right)

Definition Let T be a tree with size n, with vertices ordered in depth-first search order: $u_0 < u_1 < \cdots < u_{n-1}$.
The Łukasiewicz path $\mathcal{W}(T) = (\mathcal{W}_0(T), \dots, \mathcal{W}_n(T))$ is defined by:

- $\mathcal{W}_0(T) = 0$
- $\mathcal{W}_{i+1}(T) = \mathcal{W}_i(T) + k_{u_i}(T) - 1$ for $0 \leq i \leq |T| - 1$.

Example



Proposition The map $\{\text{trees with } n \text{ vertices}\} \longrightarrow \overline{S}_n$

$$T \longmapsto (k_{u_i}(T) - 1 : 0 \leq i \leq n-1)$$

is a bijection, where $\overline{S}_n = \{ (x_1, \dots, x_n) \in \{-1, 0, 1, \dots\}^n : x_1 + \dots + x_n = -1, \ x_1 + \dots + x_i > -1 \text{ for } 1 \leq i \leq n \}$

This can be readily shown by induction. The complete proof is a bit tedious to write and it is skipped here (the reader should convince him/her that this is true)

2) Connection with conditioned random walks

Let $\mu = (\mu(i) : i \geq 0)$ be a probability distribution on $\mathbb{Z}_+$, called the offspring distribution.

Assume that $\mu(1) \neq 1$ and $\sum_{i=0}^{\infty} i \mu(i) \leq 1$

Let $\mathbb{T}$ be the set of all finite trees ($\mathbb{T}$ is countable)

Recall that for $T \in \mathbb{T}$, $\mathbb{P}_\mu(T) \overset{(*)}{=} \prod_{u \in T} \mu(k_u(T))$ defines a probability measure on $\mathbb{T}$.

A B_μ random tree will be a $\mathbb{T}$-valued random variable, with law $\mathbb{P}_\mu$.

To make the connection with the Lukasiewicz path, we introduce the random walk $(W_n)_{n \geq 0}$: Let $(X_i)_{i \geq 1}$ be iid random variables with law $\mathbb{P}(X_1 = k) = \mu(k+1)$ for $k \geq -1$.

Set $W_0 = 0$ and $W_n = X_1 + \dots + X_n$ for $n \geq 1$.

Also set $\zeta = \inf\{k \geq 1 : W_k = -1\} \in \mathbb{N} \cup \{+\infty\}$

Important remark Observe that $\mathbb{E}[X_1] = \sum_{k \geq -1} k \mu(k+1) = \sum_{k=1}^{\infty} k \mu(k) - 1$ In particular, critical trees (for which $\sum_{k=1}^{\infty} k \mu(k) = 1$) play a special role: we have $\mathbb{E}[X_1] = 0$ iff μ is critical.

Proposition Let τ be a B_μ random tree. Then

$$(\mathcal{W}_0(\tau), \dots, \mathcal{W}_{|\tau|}(\tau)) \overset{\text{law}}{=} (W_0, \dots, W_\zeta)$$

In particular $\zeta < \infty$ a.s.

The proof is straightforward using (*) by computing the probability that the 2 random vectors are equal to $(w_0, \dots, w_n)$.

In the sequel, τ_n denotes a BP_μ random tree conditioned on having n vertices (we implicitly restrict to values of n such that $P(|\tau|=n)>0$).

Corollary
- $|\tau| \overset{law}{=} \zeta$
- $(\mathcal{W}_0(\tau_n), \dots, \mathcal{W}_n(\tau_n)) \overset{law}{=} (W_0, \dots, W_n)$ under $P(\cdot \mid \zeta = n)$

The main difficulty is that this conditioning is "non local". To make it "local" we we are going to use the so-called cycle lemma.

3) The Vervaat transform

We first introduce some notation.

Set $S_n = \{ (x_1, \dots, x_n) \in \{-1, 0, 1, \dots\}^n : x_1 + \dots + x_n = -1 \}$.

Recall that $\overline{S}_n = \{ (x_1, \dots, x_n) \in \{-1, 0, 1, \dots\}^n : x_1 + \dots + x_n = -1$ and $x_1 + \dots + x_i > -1$ for $1 \leq i \leq n \}$

We identify $\mathbb{Z}/n\mathbb{Z}$ with $\{0, 1, \dots, n-1\}$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, we define $x^{(i)} = (x_{i+1}, x_{i+2}, \dots, x_{i+n})$ with addition considered modulo n.

Example $x = (1, -1, -1, 1, -1, 2, -1, -1)$, $x^{(5)} = (2, -1, -1, 1, -1, -1, 1, -1)$.

Walks with jumps given by x and $x^{(5)}$: